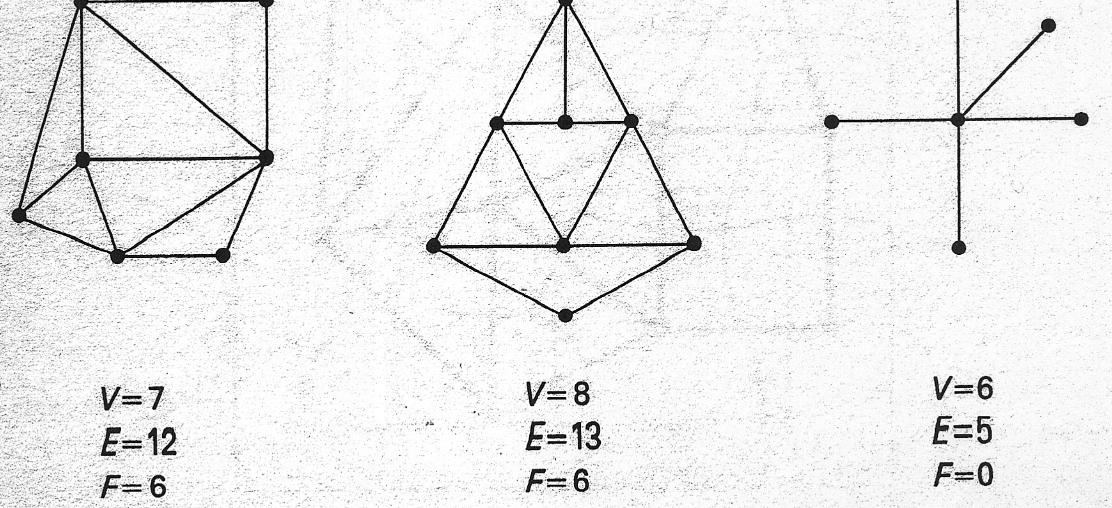
Euler's Formula

A particularly useful observation concerning networks was made (in a slightly different context) by Leonhard Euler. First a little terminology (whose origin will be explained presently). For a start, let us agree that we shall consider only networks which have the property that it is possible to get from any one node to any other by following a route consisting entirely of path connections of the network. This excludes 'pathological' examples of 'networks' such as a set of nodes with no connecting paths, but includes all the networks we shall need for our study of map colouring. Any such network divides the part of the plane which it occupies into a number of regions; these regions are called *faces*. The nodes of the network are sometimes (and in particular in connection with Euler's formula) called *vertices* of the network. The paths which connect these vertices are called *edges*.

At this stage you should draw a number of networks, and for each one tabulate the number (V) of vertices, the number (E) of edges, and the number (F) of faces, as in Figure 38. When you have done this, work out

Figure 38. Euler's formula. For any network, the number (V) of vertices, the number (E) of edges, and the number (F) of faces are such that $V - E + F = 1$.



the quantity $V - E + F$ in each case – you will find that the result is always 1. The fact that the equation

$$V - E + F = 1$$

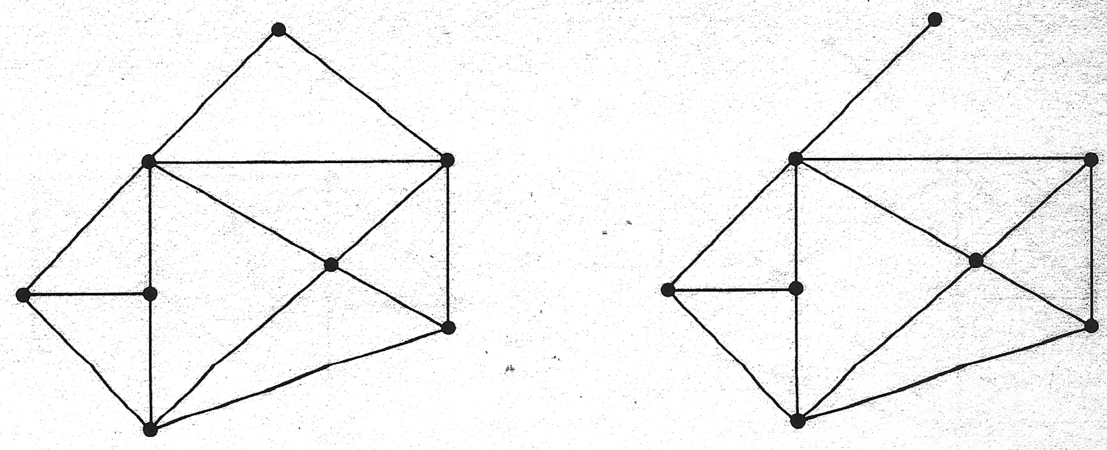
holds for any network was first proved by Euler himself.

In fact Euler was concerned not so much with networks as with polyhedra, which explains the use of the words 'vertex', 'edge', and 'face'. For any polyhedron you find that

$$V - E + F = 2$$

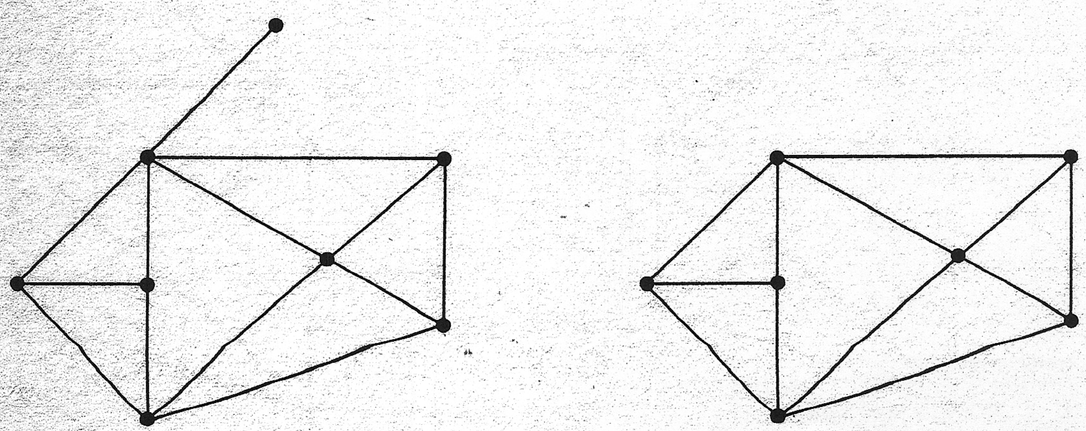
(where V, E, and F have their obvious meanings in terms of a polyhedron). To see that this is essentially the same result as that stated a moment ago for networks, notice that if you remove one face from a polyhedron and then 'stretch out' the remaining figure to lie in a plane, the former edges of the polyhedron will form a network connecting the former vertices (the nodes of the new network), and conversely if you take any network you can 'bend it round' into the shape of a polyhedron with one face missing. It is of course the removed or missing face of the polyhedron that accounts for the difference between the formula for the network and the one for the polyhedron.

Figure 39. Removal of an outside edge from a network decreases both E and F by 1, but leaves V unaltered. This does not affect the value of the quantity $V - E + F$.



The proof of Euler's network formula provides an excellent example of the kind of argument used in both graph theory and the study of the four-colour problem. Suppose you start with some network and you wish to prove that $V - E + F = 1$. What happens if you remove an outside edge of the network (assuming there is one)? Then E decreases by 1 and so does F, whilst V remains the same (see Figure 39). So the quantity $V - E + F$ remains unaltered by this action. Again, what happens if the network has a 'dangling' vertex (see Figure 40) and you remove both the vertex and the edge leading to it? Then V decreases by 1 and so does E, whilst F is unaltered, so in this case too the quantity $V - E + F$ is unchanged. Now, if you start with your given network and, like the sea eroding an island, keep on removing outside edges and dangling vertices whenever possible, you will eventually end up with just a single vertex. That is, you will have reduced your original network to a trivial one in which $V = 1$, $E = 0$, and $F = 0$. In this final 'network' the quantity $V - E + F$ is equal to 1. But none of the 'erosion' operations you performed altered $V - E + F$. So the value of this expression *in the original network* must have been 1 as well. And that proves the result. If you wish, you can try this out for yourself. Start with some arbitrarily drawn network and keep on removing outside edges and dangling vertices, tabulating the values of V, E, F, and $V - E + F$ as you go.

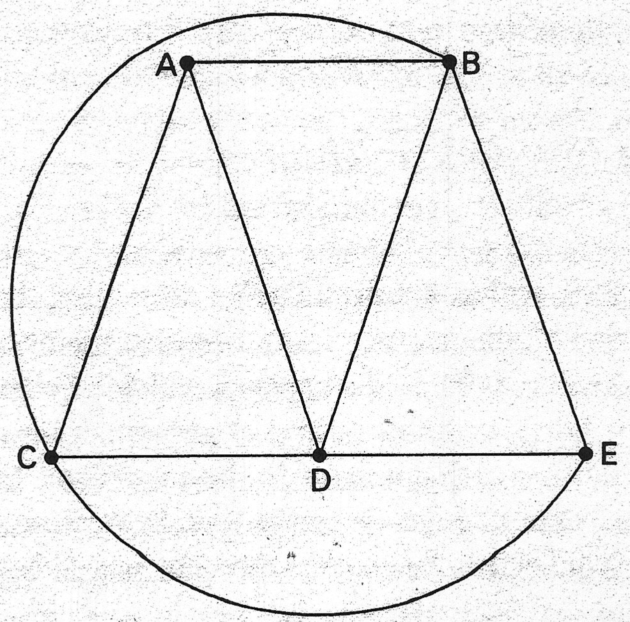
Figure 40. Removal of a 'dangling' vertex from a network decreases both V and E by 1, but leaves F unaltered. This does not affect the value of the quantity $V - E + F$.



de Morgan's Theorem

The one positive result that de Morgan managed to obtain on Guthrie's problem was to prove that in no map can the situation occur where each of five countries borders the other four. By using the neighbouring network together with Euler's formula, this is quite easy to demonstrate. In terms of networks, what de Morgan's result amounts to is that it is not possible to draw a network with five vertices so that each vertex is connected to the other four. Certainly if you try to do this (see Figure 41) you will invariably find that you are left with two vertices which cannot be connected without crossing over a path that has already been drawn, but this does not constitute a proof, since it may just be that you have drawn the earlier connections inappropriately. The following argument does not depend on a particular drawing, and thus provides a rigorous proof.

Figure 41. It is not possible to draw a network with five vertices such that each vertex is connected to the other four. No matter how you try to join up the vertices, you will be left with two vertices (A and E in the network shown) that cannot be joined without crossing one of the lines already drawn.



Suppose that it were possible to draw a network with five vertices such that each vertex is connected to all the others. If the area surrounding the network is regarded as an additional 'face', then each edge of the network will separate two faces. Moreover, since there is now one extra face, Euler's formula becomes

$$V - E + F = 2.$$

We know the value of V here, it is 5. Also, since every vertex is joined by an edge to every other, $E = 10$. (Check this for yourself!) So F has to be 7 if the above Euler formula is to hold.

So far so good. Now we perform another calculation. Since each face will be surrounded by at least three edges (this is true also for the new 'face' we introduced a moment ago, though you have to understand the word 'surrounded' is used in a topological sense in this case), counting edges by faces gives at least $3 \times 7 = 21$ edges. But if you count edges in this way (by faces), each edge will be counted twice, since it separates two faces. So the correct answer is that there must be at least $\frac{1}{2} \times 21 = 10\frac{1}{2}$ edges, which (since there is no such thing as half an edge) means that there are at least 11 edges. But, as we noted above, $E = 10$. So we have arrived at a contradictory situation, and as usual in such arguments the conclusion is that the original assumption of the argument must be false – that is, it is *not* possible to draw a network with five vertices so that each vertex is connected to all the others. That proves de Morgan's theorem about maps.

The Five-Colour Theorem

In 1879, within a year of Cayley presenting the four-colour problem to the London Mathematical Society, one of its members, a barrister called Alfred Bray Kempe, published a paper in which he claimed to prove the conjecture. But he was mistaken, and eleven years later Percy John Heawood pointed out a significant error in the argument. Heawood was however able to salvage enough to prove that five colours are always adequate, and the proof of this 'five-colour theorem' is sufficiently simple to give here.