Suppose that it were possible to draw a network with five vertices such that each vertex is connected to all the others. If the area surrounding the network is regarded as an additional 'face', then each edge of the network will separate two faces. Moreover, since there is now one extra face, Euler's formula becomes

$$V - E + F = 2.$$

We know the value of V here, it is 5. Also, since every vertex is joined by an edge to every other, $E = 10$. (Check this for yourself!) So F has to be 7 if the above Euler formula is to hold.

So far so good. Now we perform another calculation. Since each face will be surrounded by at least three edges (this is true also for the new 'face' we introduced a moment ago, though you have to understand the word 'surrounded' is used in a topological sense in this case), counting edges by faces gives at least $3 \times 7 = 21$ edges. But if you count edges in this way (by faces), each edge will be counted twice, since it separates two faces. So the correct answer is that there must be at least $\frac{1}{2} \times 21 = 10\frac{1}{2}$ edges, which (since there is no such thing as half an edge) means that there are at least 11 edges. But, as we noted above, $E = 10$. So we have arrived at a contradictory situation, and as usual in such arguments the conclusion is that the original assumption of the argument must be false – that is, it is *not* possible to draw a network with five vertices so that each vertex is connected to all the others. That proves de Morgan's theorem about maps.

The Five-Colour Theorem

In 1879, within a year of Cayley presenting the four-colour problem to the London Mathematical Society, one of its members, a barrister called Alfred Bray Kempe, published a paper in which he claimed to prove the conjecture. But he was mistaken, and eleven years later Percy John Heawood pointed out a significant error in the argument. Heawood was however able to salvage enough to prove that five colours are always adequate, and the proof of this 'five-colour theorem' is sufficiently simple to give here.

First of all, notice that by reasoning as we did above to relate the Euler formula for networks to that for polyhedra, we may conclude that it does not matter (as far as the four- or the five-colour theorem is concerned) whether we draw our maps on the plane or on the surface of a sphere. If we start with a map on a sphere we can deform it to an equivalent map on the plane by piercing a hole in the middle of one of the regions and pulling the entire map out flat (so that the pierced region becomes one which surrounds the rest of the map). Conversely, if we are given a map on the plane we may regard the region surrounding the map as an extra country, and fold the entire map round into the shape of a sphere (bringing the added surrounding region together to form an 'enclosed' region just like all the others). This procedure shows that if every planar map can be coloured with N colours, so can every map on the sphere, and vice versa.

We shall in fact prove the 'five-colour theorem' for maps drawn on a sphere. We shall make use of the Euler formula, which for maps on a sphere is

$$V - E + F = 2.$$

Our use of this formula will be in connection with the map itself, rather than with the associated neighbouring network as was the case with the proof of de Morgan's theorem. (So a *face* will be a region of the map, an *edge* will be a border, and a *vertex* will just be a point where three or more borders meet.)

The idea of the proof is to start with a given (entirely arbitrary) map drawn on a sphere, and gradually modify it by a process of merging two or more adjacent countries into one, so that eventually a map is obtained having at most five countries – which can obviously be coloured using five colours or fewer. Provided the steps used in the modification process do not reduce the number of colours required to colour the map, this will prove that five colours suffice for the original map. So the crux of the proof is to describe the individual processes which are used to *reduce* a given map to a simpler one (i.e. one with fewer countries) without reducing the number of colours necessary to colour the map. There are six different *reduction processes*, each of which is applicable in a different situation depending on particular configurations of countries on the map.

First of all, if one region is entirely surrounded by another (see Figure 42(i)), then the inner region may be merged with the surrounding one. Any colouring of the new map using at least two colours can be extended to a colouring of the original map using the same colours: the inner region is simply assigned a colour other than the one used to colour the entire merged region on the modified map.

The next reduction operation applies whenever there is a vertex at which more than three regions touch. For if at least four regions touch, then (and you may need to think about this for a moment) one pair of these regions will not have a common border (anywhere on the map!) and these two regions can be merged into one (see Figure 42(ii)). Given any colouring of the modified map, the original map may be coloured using the same number of colours by assigning the same colour to the two regions that were merged, and colouring the rest of the map the same in both cases. By applying this reduction repeatedly the map can be modified so that only three regions touch at each vertex. This will be assumed to be the case for the remainder of the reductions.

If there is a region which borders on just two others (see Figure 42(iii)), then that region may be 'removed' by merging it with one of these two. If the new map can be coloured using at least three colours, the original map can be coloured using the same colours simply by colouring the merged central region differently from the two surrounding areas.

Any region having three neighbours can be 'removed' by merging it with one of its neighbours (see Figure 42(iv)), and as in the previous case if the new map can be coloured using at least four colours then the original can be coloured using the same colours.

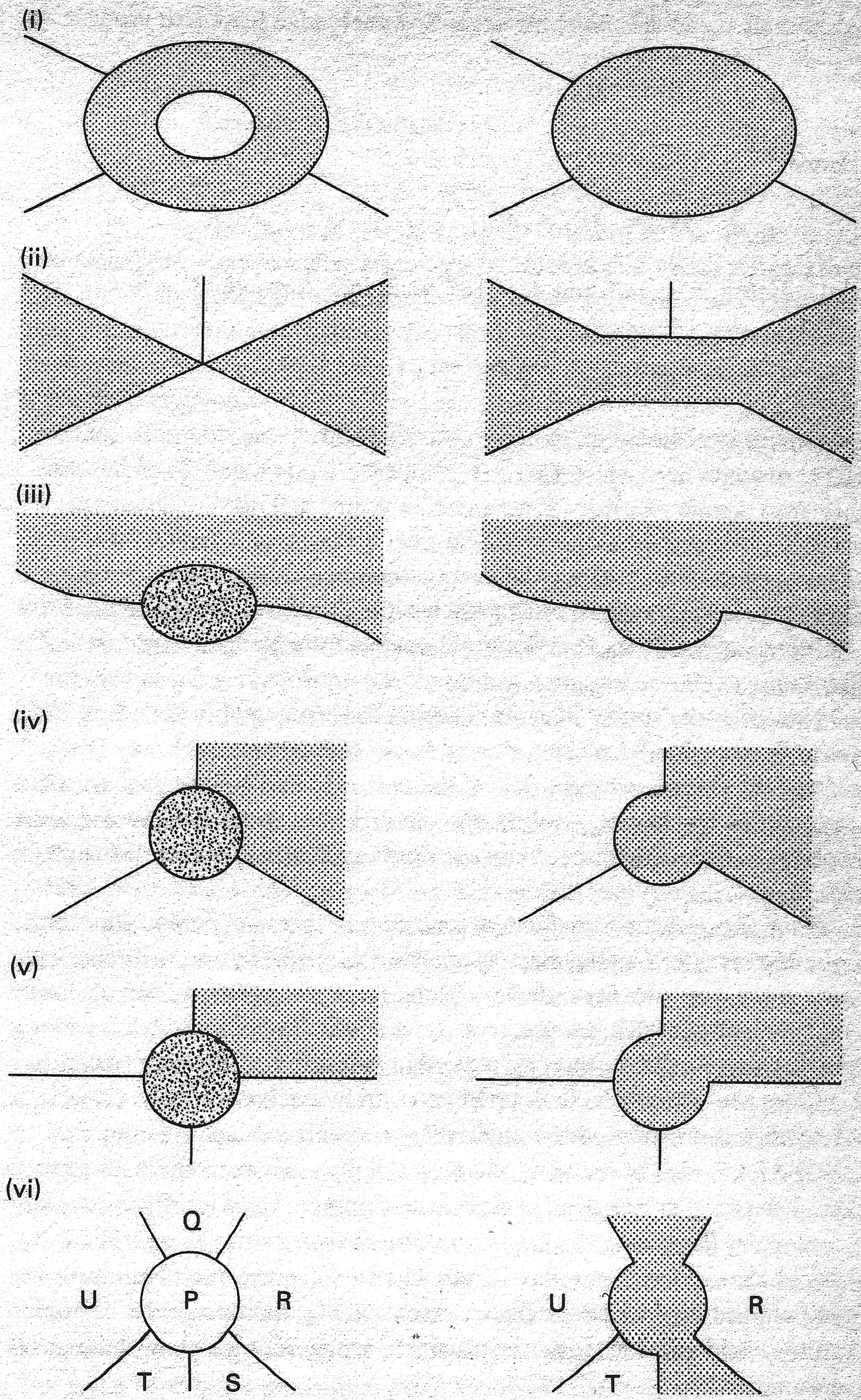
Likewise any region having four neighbours can be merged with one of its neighbours (see Figure 42(v)), and this will not involve any change in the colour requirements when five colours are available.

By applying the above reduction procedures as often as possible, you will end up with a map in which no region is surrounded by another, in which each vertex lies on exactly three edges, and all of whose regions have at least five edges. In fact, at least one region will have exactly five edges, as we now prove.

There are V vertices, E edges, and F regions. Let a be the average of the number of edges bordering each region. (So a might be fractional.) Since each edge lies between two regions,

$$2E = aF.$$

Figure 42 (facing). Reductions used in the proof of the five-colour theorem (see the text for details).


(i)
(ii)
(iii)
(iv)
(v)
(vi)
Q
U
P
R
T
S
U
R
T

Also, each vertex lies on three edges and each edge joins two vertices, so

$$3V = 2E.$$

Hence

$$3V = 2E = aF.$$

Substituting $V = \frac{1}{3}aF$ and $E = \frac{1}{2}aF$ in Euler's formula $V - E + F = 2$ gives

$$\tfrac{1}{3}aF - \tfrac{1}{2}aF + F = 2,$$

so

$$a = 6 - \frac{12}{F}.$$

Thus a is less than 6. Since the average number of edges for each region is less than 6, some regions must have less than 6 edges. But all regions have at least 5 edges. So some regions will have exactly 5 edges, as we set out to prove.

Now consider such a five-edged region P, with neighbours Q, R, S, T, U, as in Figure 42(vi). One pair of neighbours of P do not touch, say Q and S. Merge the three regions P, Q, S. If the new map can be coloured with five colours, so can the original. In the merged map Q and S have the same colour, so there are four colours surrounding P, which leaves one to spare for P.

That completes the reduction procedure. Since each step reduces the number of regions on the map, by applying it repeatedly you will eventually arrive at a map with five or fewer regions. Since any such map can obviously be coloured with five colours, so can the original map. Indeed, by working back through the various reductions a colouring of the map using five colours can actually be built up in an entirely mechanical way. (Try it out for yourself, starting with a moderately complicated-looking map.)

The basic idea of reducing the map step by step, as in the above proof, has been used in practically every serious attempt to prove the four-colour conjecture (including the final successful attempt) since Kempe first introduced the device in his flawed attempt to solve the problem. Since the actual solution was essentially an extension of what Kempe did (though a quite considerable extension), it is worth while taking a closer look at his argument.