THE MILLENIUM PROBLEMS
The Seven Greatest Unsolved Mathematifcal Puzzles of our Time

Freshman Seminar
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Fall 2014

Preface

In May 2000, at a highly publicized meeting in Paris, the Clay Mathematics Institute announced that seven $1 million prizes were being offered for the solutions to each of seven unsolved problems of mathematics —problems that an international committee of mathematicians had judged to be the seven most difficult and most important in the field today.

For some of these problems, it takes considerable effort simply to understand the individual terms that appear in the statement of the problem. It is not possible to describe most of them accurately in lay terms—or even in terms familiar to someone with a university degree in mathematics.

The goal of this seminar is to provide the background to each problem, to describe how it arose, explain what makes it particularly difficult, and give some sense of why mathematicians regard it as important.

This seminar is not for those who want to tackle one of the problems, but rather for mathematicians and non-mathematicians alike who are curious about the current state at the frontiers of humankind's oldest body of scientific knowledge.

After three thousand years of intellectual development, what are the limits of our mathematical knowledge?

To benefit from this seminar, all you need by way of background is a good high school knowledge of mathematics. But on its own, that won't be enough. You will also need sufficient interest in the subject.

The Millennium Problems are the hardest and most important unsolved mathematics problems in the world; they have resisted numerous attempts at solution, over many years, by the best mathematical minds around.

Even achieving a layperson's appreciation of what they are about takes considerable effort. Aren't all pinnacles of human achievement of interest?

Go to www.claymath.org for a 20 minute film containing an introductory account of the problems. NOTE: YOU WON'T FIND THIS FILM ANYWHERE ON THE WEB, BUT THE WEBSITE SHOULD STILL BE OF INTEREST.

Chapter 0. The Gauntlet is Thrown

"It is the desire for truth and the response to the beauty and elegance of mathematics that drives mathematicians." —LANDON CLAY, benefactor of the Clay Millennium Problems.

One year before the 2000 meeting in Paris, Clay had established the Clay Mathematics Institute (CMI), a nonprofit organization based in his home town of Cambridge, Massachusetts, designed to promote and support mathematical research.

Though not a mathematician (he majored in English at Harvard), Clay also funded a chair (in Mathematics) at his alma mater, and then the Millennium Prizes. These initiatives were partly a response to what he saw as low public funding for an important subject.

By offering a major cash prize and inviting the world's press to the Paris meeting, Clay ensured that the Millennium Problems—and mathematics in general—would get international media attention. (Why travel to Paris?)

Exactly 100 years earlier, in 1900, Paris had been host to a similar event. On August 8, 1900, the German mathematician David Hilbert, an international leader in the field, gave an invited address in which he laid out an agenda for mathematics for the twentieth century: The (23) **Hilbert Problems**.

Some were easier than anticipated and soon solved; others were two imprecise to admit a definite answer. Those that remained turned out to be mathematical problems of great difficulty, and became beacons guiding mathematicians forward into the future.

Solving any of these brought the solver instant fame among the mathematical community, every bit as significant as the award of a Nobel Prize.

By 2000, all but one of the genuine Hilbert Problems had been solved, and the time was ripe for mathematicians once again to take stock.

What were the gold-ring problems at the end of the Second Millennium—the unsolved problems that everyone agreed were the Mount Everests of mathematics?

The (names of the) Problems

The Riemann Hypothesis—Chapter 1

(distribution of prime numbers)

Yang-Mills and the Mass Gap Hypothesis—Chapter 2

(quantum physics; why electrons have mass)

P versus NP—Chapter 3

(how efficiently can computers solve problems)

The Navier-Stokes Equations—Chapter 4

(the motion of fluids and gases; nautical and aeronautical engineering)

The Poincare Conjecture—Chapter 5

(how do you distinguish an apple from a doughnut?—in four dimensions!)

The Birch and Swinnerton-Dyer Conjecture—Chapter 6

(solutions of algebraic equations in whole numbers; —Pythagorean triples (Euclid) and Fermat's Last Theorem (Wiles))

The Hodge Conjecture—Chapter 7

(approximating the shape of given object by gluing together simple geometric building blocks of increasing dimension)

Why are the problems so hard to understand?

"A mathematician is the only scientist who can legitimately claim, 'I lie down on the couch, close my eyes, and work'."—RONALD GRAHAM, UC San Diego

Mathematics is almost entirely cerebral—it goes on in the brain, without any direct connection to something in the physical world.

In physics or chemistry or biology, the object of the scientist's thought is generally some phenomenon in her physical world. This makes it easier to explain.

Mathematics does not have this helpful link to reality and must use words instead. How can the non mathematical person understand those words, when they in turn don't link to anything in everyday experience.

Two and a half thousand years ago, a young follower of Pythagoras proved that the square root of 2 is not a rational number, that is, cannot be expressed as a fraction.

This discovery came as such a shock to the Pythagoreans that their progress in mathematics came to a virtual halt. Only toward the end of the nineteenth century did mathematicians finally work out a rigorous theory of real numbers.

These days, even many non mathematicians feel comfortable using real numbers, complex numbers, and negative numbers.

Similarly, in geometry, the discovery in the eighteenth century that there were other geometries besides the one that Euclid had descried in his famous book *Elements* caused both the experts and the non mathematicians enormous conceptual problems.

Most mathematical concepts are built up not from everyday phenomena but from earlier mathematical concepts.

Who (Just) wants to be a millionaire?

Will the prospect of winning a million-dollar prize have any real effect on whether one of the Millennium Problems is solved? The answer is definitely NO.

To solve one of these problems, you would almost certainly need a Ph. D. in mathematics, be good enough to get a tenured position at one of the top universities in the world, and be prepared to devote many years to an in-depth study of the relevant area(s).

Anyone who needs the prospect of a million-dollar prize to persuade him or her to do that with their life simply doesn't have the requisite commitment to mathematical research.

Mathematicians are no different from Olympic athletes, who invariably value gold medals far more than lucrative advertising and merchandising contracts.

Ultimately, mathematicians pursue these problems for the same reason the famous British mountaineer George Mallory gave in answer to the newspaper reporter's question, "Why do you want to climb Mount Everest?": "Because it is there."

Mallory's words reveal a fundamental part of the human spirit: the urge to break new ground, to run faster, jump higher, or climb higher than anyone before. That is why the Millennium Problems are the Mount Everests of mathematics.

It was not a masochistic streak or a macabre death wish that led mountain climbers to endure enormous hardship and risk their lives trying to scale tall mountains. On the contrary, it was a powerful love of life that drove them to do what they did. Similarly for mathematics.

The world's very best mathematicians, the kinds of people who might stand a chance of solving a Millennium Problem, simply bring to the subject a greater degree of commitment and passion than do those of us who gain our intellectual kicks from wrestling with the lesser challenges of mathematics.

The seven Millennium Problems are the current Mount Everests of mathematics. What you will be able to see from any one of those seven summit is hard to say with any precision. There can be no doubt, however, that if any one of the Problems is solved, we will be able to see so far that the world cannot possibly be left unchanged. That is the true prize, $1 million notwithstanding.