THE POINCARÉ CONJECTURE
The Mathematics of Smooth Behavior

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Henri Poincaré

Henri Poincaré (1854-1912) was one of the greatest and most innovative mathematicians and physicists the world has ever seen. He came very close to beating Einstein to the discovery of special relativity.

A world renowned expositor of science, he excelled at linking together new ideas.

Despite an interest in all aspects of mining that would continue throughout his life, by 1875 he knew that his main passion was mathematics

The wide range of his knowledge and his ability to see connections between seemingly very distinct areas allowed him to attack problems from many different and often novel angles.

It is in Poincaré's work in the branch of mathematics called *topology* in which the sixth millennium problem arises: the Poincaré conjecture.

Rubber-sheet geometry

The standard map of the London Underground illustrates the immense power of topology, despite being completely inaccurate in every respect but two. For example, distances are wrong and the straight lines are not really straight.

If a station is shown to be north of the River Thames, then the real station is indeed north of the Thames. The order in which the stations lie on each line and the places (stations) where any two lines intersect, is accurately depicted.

This example and another one (wiring diagram) illustrate the essence of two-dimensional topology, If the Underground map were printed on a perfectly elastic sheet of rubber, it could be stretched and compressed but every detail would remain intact.

The configuration of a network (points connected by various lines) is a *topological property*. To change the network, you must either break a connection or add one.

As long as the map is topologically accurate, the exact design does not matter.

Topology

The development of topology was not driven by the needs of any area of applied mathematics, but rather it came from within pure mathematics itself, from the struggle to understand why the differential calculus worked.

From the moment Newton and Leibniz introduced calculus in the middle of the 17th century, it was used extensively, but no one really understood why.

The drive for ever more detailed analysis (real numbers, infinite processes, mathematical reasoning) was accompanied by a dramatic increase in abstraction. Until the middle of the 19th century mathematics dealt with objects and patterns that came from our everyday experience.

The 19th century saw the appearance of a host of new kinds of objects and patterns, not recognized as part of everyday experience: geometries in which parallel lines meet (non-Euclidean geometries), geometries of 4 or more dimensions, algebra where the symbols stand for symmetries in figures (group theory) or logical thoughts (propositional logic). The development of topology was part of this proliferation of new abstractions.

The idea behind topology was to develop a "geometry" that studies properties of figures that are not destroyed by continuous deformation, and thus do not depend on notions such as straight lines, circles, cubes, and so on, or on measurement of length, areas, volumes, or angles.

The connection between topology and calculus is that both depend on being able to handle the infinitely small. Two points that are "infinitely close" remain infinitely close after being transformed by a "topological transformation."

Neither stretching, compressing, nor twisting a rubber sheet of rubber destroys closeness. The only way to destroy the closeness is to cut or tear the sheet—a forbidden operation in topology.

The development of topology put calculus on a strong footing. What else is it good for? There's more there than you'd think!

For example, topology forms the mathematical foundation of **superstring theory**, physicists' most recent theory of the nature of the universe.

In 1735, Euler solved a long-standing puzzle called the *Königsberg bridges problem*

Although Euler solved one of the first topological puzzles and proved one of the first topological theorems ($V - E + F = 1$), topology did not really get under way until the late 19th century, when Poincaré came on the scene.

In topology, we study properties of figures and objects that remain unchanged when the figure or object is subjected to a continuous deformation. Continuous means that the deformation does not involve any cutting, tearing, or gluing.

Examples: football = soccer ball = tennis ball; coffee cup = doughnut

Why is a ball topologically distinct from a torus (doughnut)? Simply not being able to find a continuous transformation that turns one object into another does not provide conclusive proof that the two objects are topologically different.

How do you know for sure that a ball is topologically distinct from a torus? Answer: they have different *Euler characteristics*.

Euler's network formula $V - E + F = 1$ can be restated as "The Euler characteristic of a network in a plane is 1".

The Euler characteristic of a network in a plane is 1
The Euler characteristic of a network on a sphere is 2
The Euler characteristic of a network on a torus is 0
The Euler characteristic of a network on a double-ringed torus is -2
The Euler characteristic of a network on a Klein bottle is 0

What the heck is a Klein bottle?

The Euler characteristic of a network does not change if the network is subjected to a continuous transformation.

Conlusion: We know without a doubt (reasonable doubt?) that a two-dimensional plane, the surface of a sphere, the surface of a torus, and the surface of a double-ringed torus are all topologically different.

The Four Color Theorem is a topological result, since continuous deformation of the sheet on which the map is drawn will not alter the pattern of shared borders

The Four Color Theorem is about maps drawn in a plane. What about maps drawn on a sphere, torus, a double-ringed torus?

The *chromatic number* of a surface is the least number of colors you need to be able to color any map drawn on that surface.

The chromatic number of a map in a plane is 4
The chromatic number of a map on a sphere is 4
The chromatic number of a map on a torus is 7
The chromatic number of a map on a double-ringed torus is ?
The chromatic number of a map on a Klein bottle is 6

What the heck is a Klein bottle?

Surely, any surface has two sides, right? **WRONG!**

A surface with only one side (and only one edge): a *Möbius band*.

The closed surface (no sides) that corresponds to a Möbius band: *Klein bottle*

Despite You-Tube, the Klein bottle exists (as a mathematical object) only in 4 dimensional space.

The topological property of a Klein bottle that corresponds to the one-sided nature of its surface is a concept called *nonorientability*.

A sphere (or a plane) is *orientable* and so are a torus and a double torus. The Klein bottle (and its ilk) are called nonorientable.

Classification theorem for two-dimensional surfaces

If two surfaces have the same Euler characteristic and are either both orientable or both nonorientable, then they are in fact the same
(except for continuous transformations)

This is proved by performing "surgery" on a sphere (the most aesthetically perfect closed surface)

The term surgery is apt, since a typical operation in constructing a given surface starting with a sphere involves cutting one or more pieces from the sphere, twisting, turning, stretching, or shrinking each of those pieces, and then sewing those pieces back into the sphere again.

The classification theorem tells us that
• any orientable surface is topologically equivalent to a sphere with a certain number of "handles" sewn onto it.
• any nonorientable surface is topologically equivalent to a sphere with a certain number of "crosscaps" sewn in.

In the early years of the twentieth century, Poincaré and other mathematicians set out to classify higher-dimensional analogues of surfaces, called “manifolds.”

A 1-manifold is a circle (or a line) in two dimensions
A 2-manifold is a surface in three dimensions
A 3-manifold is a “hypersurface” in four dimensions
An n-manifold “lives” in $(n+1)$-dimensional space.

A natural first step in this endeavor was to look for a simple topological property that tells you when a given hypersurface is topologically equivalent to a sphere.

In the case of two dimensional surfaces, there is such a property: every loop can be shrunk to a point without leaving the surface.
(The term of art for this is *simply connected*)

Poincaré Conjecture: Is the same true for a three-dimensional hypersphere?

The mathematical theories of matter that physicists are currently working on view the universe we live in as having 11 dimensions. There is no mathematical reason to stop at three dimensions.

Poincaré suggested to classify manifolds of three and more dimensions by taking a "sphere" of the respective dimension as a base figure and then applying surgery.

Surgery on the two dimensional sphere (which lives in 3 dimensional space) had to be replaced by surgery on a three-dimensional sphere (which lives in 4 dimensional space—hypersphere?). An n-sphere lives in $n+1$ dimensional space.

In 1960, this method was used by Stephen Smale, to prove that, for $n \geq 5$, every simply connected n-manifold was equivalent to the n-sphere

In 1981, totally new methods were used by Michael Freedman to prove that every simply connected 4-manifold was equivalent to the 4-sphere

Smale and Freedman were both awarded a Fields Medal (mathematical equivalent of the Nobel Prize) for these accomplishments.

Thus, at the time the Millennium Problem Prizes were announced (2000), the Poincaré conjecture had been shown to be true for every dimension except three—the very dimension for which Poincaré originally raised the question.