APPENDIX 3

HOW EULER DISCOVERED THE ZETA FUNCTION

Now that we know about infinite sums, we can ask how Euler discovered that one particular infinite sum, the zeta function, provides information about the pattern of the primes. Besides being a fascinating question in its own right, the answer will help you to follow the account of the Birch and Swinnerton-Dyer Conjecture in Chapter 6.

Knowing that the harmonic series has an infinite sum, Euler wondered about the "prime harmonic series"

$$PH = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$$

which you get by adding the reciprocals of all the primes. Is its sum finite or infinite?

He began by regarding PH as a subseries of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$$

This series has an infinite sum, so it did not allow Euler to do what he wanted to do next. He therefore looked instead at the related sum

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

you get by raising each term in the harmonic series to the power s. Provided s is bigger than 1, this sum is finite, and so you can split it up into two parts, the first part being all the prime terms, the second all the nonprime terms, like this:

$$\zeta(s) = \left[1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \cdots\right] + \left[\frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{9^s} + \cdots\right]$$

The idea then is to show that if you were to take s closer and closer to 1, the first sum

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \cdots$$

increases without bound, and hence that taking $s = 1$,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots$$

is infinite.

A key step in this argument was to establish the celebrated equation

$$\zeta(s) = \frac{1}{1-(1/2^s)} \times \frac{1}{1-(1/3^s)} \times \frac{1}{1-(1/5^s)} \times \frac{1}{1-(1/7^s)} \times \cdots$$

where the product on the right is taken over all terms $\dfrac{1}{1-(1/p^s)}$ where p is a prime. Euler's idea was to start with the formula for the geometric series we met a moment ago:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad (0 < x < 1)$$

For any prime number p and any $s > 1$, we can set $x = 1/p^s$ to give

$$\frac{1}{1-(1/p^s)} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots$$

The expression on the left is a typical term in Euler's infinite product, of course, so the above equation provides an infinite sum expression for each term in the infinite product. What Euler did next was multiply together all of these infinite sums to give an alternative expression for his infinite product. Using the ordinary algebraic rules for multiplying (a finite number of finite) sums, but applying them this time to an infinite number of infinite sums, you see that when you write out the right-hand side of the product as a single infinite sum, its terms are all the expressions of the form

$$\frac{1}{p_1^{k_1 s} \cdots p_n^{k_n s}}$$

where $p_1, \ldots, p_n$ are different primes and $k_1, \ldots, k_n$ are positive integers, and each such combination occurs exactly once. But by the fundamental theorem of arithmetic, every positive integer can be expressed in the form $p_1^{k_1 s} \cdots p_n^{k_n s}$. Hence the right-hand side of the product is just a rearrangement of the sum

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

i.e., $\zeta(s)$. (You have to be a bit careful how you do this, to avoid getting into difficulties with infinities. The details are not particularly difficult, but it would take us too far from our path to give the complete argument.)

Now, from our point of view—and indeed from the point of view of the subsequent development of mathematics—it was not so much the fact that the prime harmonic series has an infinite sum that is important, even though it did provide a completely new proof of Euclid's result that there are infinitely many primes. Rather, Euler's infinite product formula for $\zeta(s)$ marked the beginning of analytic number theory.

In 1837, the German mathematician Peter Gustav Lejeune Dirichlet generalized Euler's method to prove that in any arithmetic progression $a, a+k, a+2k, a+3k, \ldots$, where a and k have no common factor, there are infinitely many primes. (Euclid's theorem can be regarded as the special case of this for the arithmetic progression $1, 3, 5, 7, \ldots$ of all odd numbers.) The principal modification to Euler's method that Dirichlet made was to modify the zeta function so that the primes were separated into separate categories, depending on the remainder they left when divided by k. His modified zeta function had the form

$$L(s,\chi) = \frac{\chi(1)}{1^s} + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \frac{\chi(4)}{4^s} + \cdots$$

where $\chi(n)$ is a special kind of function—which Dirichlet called a "character"—that splits the primes up in the required way. In particular, it must be the case that $\chi(mn) = \chi(m)\chi(n)$ for any m, n. (The other conditions are that $\chi(n)$ depends only on the remainder you get when you divide n by k, and that $\chi(n) = 0$ if n and k have a common factor.)

Any function of the form $L(s, \chi)$ where s is a real number greater than 1 and χ is a character is known as a Dirichlet L-series. The Riemann zeta function is the special case that arises when you take $\chi(n) = 1$ for all n.

Mathematicians subsequent to Dirichlet used L-series (in the more general case where the variable s and the numbers $\chi(n)$ are allowed to be complex numbers) to prove a great many results about prime numbers, thereby demonstrating that Dirichlet's series provides an extremely powerful tool for the study of the primes.

A key result about L-series is that, as with the zeta function, they can be expressed as an infinite product over the prime numbers (sometimes known as an Euler product), namely,

$$\begin{aligned} L(x,\chi) = {} & \frac{1}{1-(\chi(2)/2^s)} \times \frac{1}{1-(\chi(3)/3^s)} \\ & \times \frac{1}{1-(\chi(5)/5^s)} \times \frac{1}{1-(\chi(7)/7^s)} \times \cdots \end{aligned}$$

(provided the real part of s is not negative), where the product is taken over all expressions of the form

$$\frac{1}{1-(\chi(p)/p^s)}$$

where p is a prime number.