diffgeom 10_12.pdf

1.4 Smooth Manifolds Defined*

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. A map $f : U \to V$ is called **smooth** iff it is infinitely differentiable, i.e. iff all its partial derivatives

$$\partial^\alpha f = \frac{\partial^{\alpha_1+\cdots+\alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad \alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}_0^n,$$

exist and are continuous. In later chapters we will sometimes write $C^\infty(U, V)$ for the set of smooth maps from U to V.

Definition 1.4.1. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. For a smooth map $f = (f_1, \dots, f_m) : U \to V$ and a point $x \in U$ the* **derivative of** f **at** x *is the linear map $df(x) : \mathbb{R}^n \to \mathbb{R}^m$ defined by*

$$df(x)\xi := \left.\frac{d}{dt}\right|_{t=0} f(x + t\xi) = \lim_{t \to 0} \frac{f(x + t\xi) - f(x)}{t}, \qquad \xi \in \mathbb{R}^k.$$

This linear map is represented by the **Jacobian matrix** *of f at x which will also be denoted by*

$$df(x) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix} \in \mathbb{R}^{m \times n}.$$

Note that we use the same notation for the Jacobian matrix and the corresponding linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

The derivative satisfies the **chain rule**. Namely, if $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, $W \subset \mathbb{R}^\ell$ are open sets and $f : U \to V$ and $g : V \to W$ are smooth maps then $g \circ f : U \to W$ is smooth and

$$d(g \circ f)(x) = dg(f(x)) \circ df(x) : \mathbb{R}^n \to \mathbb{R}^\ell \tag{1.4.1}$$

for every $x \in U$. Moreover the identity map $\mathrm{id}_U : U \to U$ is always smooth and its derivative at every point is the identity map of $\mathbb{R}^n$. This implies that, if $f : U \to V$ is a **diffeomorphism** (i.e. f is bijective and f and f^{-1} are both smooth), then its derivative at every point is an invertible linear map. This is why the Invariance of Domain Theorem (discussed after Definition 1.3.1) is easy for diffeomorphisms: if $f : U \to V$ is a diffeomorphism, then the Jacobian matrix $df(x) \in \mathbb{R}^{m \times n}$ is invertible for every $x \in U$ and so $m = n$. The Inverse Function Theorem (see Theorem A.2.2 in Appendix A.2) is a kind of converse.

Definition 1.4.2 (Smooth Manifold). *Let M be a set. A* **chart** *on M is a pair (ϕ, U) where $U \subset M$ and ϕ is a bijection from U to an open subset $\phi(U) \subset \mathbb{R}^m$ of some Euclidean space. Two charts (ϕ_1, U_1) and (ϕ_2, U_2) are said to be* **smoothly compatible** *iff $\phi_1(U_1 \cap U_2)$ and $\phi_2(U_1 \cap U_2)$ are both open in $\mathbb{R}^m$ and the* **transition map**

$$\phi_{21} = \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2) \tag{1.4.2}$$

is a diffeomorphism. A **smooth atlas** *on M is a collection $\mathscr{A}$ of charts on M any two of which are smoothly compatible and such that the sets U, as (ϕ, U) ranges over $\mathscr{A}$, cover M (i.e. for every $p \in M$ there is a chart $(\phi, U) \in \mathscr{A}$ with $p \in U$). A* **maximal smooth atlas** *is an atlas which contains every chart which is smoothly compatible with each of its members. A* **smooth manifold** *is a pair consisting of a set M and a maximal atlas $\mathscr{A}$ on M.*

Lemma 1.4.3. *If $\mathscr{A}$ is an atlas, then so is the collection $\overline{\mathscr{A}}$ of all charts compatible with each member of $\mathscr{A}$. The atlas $\overline{\mathscr{A}}$ is obviously maximal. In other words, every atlas extends uniquely to a maximal atlas.*

Proof. Let (ϕ_1, U_1) and (ϕ_2, U_2) be charts in $\overline{\mathscr{A}}$ and let $x \in \phi_1(U_1 \cap U_2)$. Choose a chart $(\phi, U) \in \mathscr{A}$ such that $\phi_1^{-1}(x) \in U$. Then $\phi_1(U \cap U_1 \cap U_2)$ is an open neighborhood of x in $\mathbb{R}^m$ and the transition maps

$$\begin{aligned}\phi \circ \phi_1^{-1} &: \phi_1(U \cap U_1 \cap U_2) \to \phi(U \cap U_1 \cap U_2),\\ \phi_2 \circ \phi^{-1} &: \phi(U \cap U_1 \cap U_2) \to \phi_2(U \cap U_1 \cap U_2)\end{aligned}$$

are smooth by definition of $\overline{\mathscr{A}}$. Hence so is their composition. This shows that the map $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is smooth near x. Since x was chosen arbitrary, this map is smooth. Apply the same argument to its inverse to deduce that it is a diffeomorphism. Thus $\overline{\mathscr{A}}$ is an atlas. □

Definitions 1.4.2 and 1.3.2 are *mutatis mutandis* the same, so every smooth atlas on a set M is *a fortiori* a topological atlas, i.e. every smooth manifold is a topological manifold. (See Lemma 1.3.3.) Moreover the definitions are worded in such a way that it is obvious that every smooth map is continuous.

Exercise 1.4.4. Show that each of the atlases from the examples in §1.2 is a smooth atlas. (You must show that the transition maps from Exercise 1.3.5 are smooth.)

When $\mathscr{A}$ is a smooth atlas on a topological manifold M one says that $\mathscr{A}$ is a **smooth structure** on the (topological) manifold M iff $\mathscr{A} \subset \mathscr{B}$, where $\mathscr{B}$ is the maximal topological atlas on M. *When no confusion can result we generally drop the notation for the maximal smooth atlas as in the following exercise.*

Exercise 1.4.5. Let M, N, and P be smooth manifolds and $f : M \to N$ and $g : N \to P$ be smooth maps. Prove that the identity map id_M is smooth and that the composition $g \circ f : M \to P$ is a smooth map. (This is of course an easy consequence of the chain rule (1.4.1).)

Remark 1.4.6. It is easy to see that a topological manifold can have many distinct smooth structures. For example, $\{(\mathrm{id}_{\mathbb{R}}, \mathbb{R})\}$ and $\{(\phi, \mathbb{R})\}$ where $\phi(x) = x^3$ are atlases on the real numbers which extend to distinct smooth structures but determine the same topology. However these two manifolds are diffeomorphic via the map $x \mapsto x^{1/3}$. In the 1950's it was proved that there are smooth manifolds which are homeomorphic but not diffeomorphic and that there are topological manifolds which admit no smooth structure. In the 1980's it was proved in dimension $m = 4$ that there are uncountably many smooth manifolds that are all homeomorphic to $\mathbb{R}^4$ but no two of them are diffeomorphic to each other. These theorems are very surprising and very deep.

A collection of sets and maps between them is called a *category* if the collection of maps contains the identity map of every set in the collection and the composition of any two maps in the collection is also in the collection. The sets are called the *objects* of the category and the maps are called the *morphisms* of the category. An invertible morphism whose inverse is also in the category is called an *isomorphism*. Some examples are the category of *all* sets and maps, the category of topological spaces and continuous maps (the isomorphisms are the homeomorphisms), the category of topological manifolds and continuous maps between them, and the category of smooth manifolds and smooth maps (the isomorphisms are the diffeomorphisms). Each of the last three categories is a subcategory of the preceding one.

Often categories are enlarged by a kind of "gluing process". For example, the "global" category of smooth manifolds and smooth maps was constructed from the "local" category of open sets in Euclidean space and smooth maps between them via the device of charts and atlases. (The chain rule shows that this local category is in fact a category.) The point of Definition 1.3.2 is to show (via Lemma 1.3.3) that topological manifolds can be defined in an manner analogous to the definition we gave for smooth manifolds in Definition 1.4.2.