diffgeom 4_7.pdf

dual of a vector space V, but when $\mathbb{K}$ is a field like $\mathbb{R}$ or $\mathbb{C}$ the notation $\mathbb{K}^*$ is sometimes used for the multiplicative group $\mathbb{K} \setminus \{0\}$. The terms smooth, infinitely differentiable, *and C^∞ are all synonymous.*

1.2 Coordinates

The rest of this chapter defines category of smooth manifolds and smooth maps between them. Before giving the precise definitions we will introduce some terminology and give some examples.

Definition 1.2.1. *A* **chart** *on a set M is a pair (ϕ, U) where U is a subset of M and $\phi : U \to \phi(U)$ is a bijection[1] from U to an open set $\phi(U)$ in $\mathbb{R}^m$. An* **atlas** *on M is a collection $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$ of charts such that the domains U_α cover M, i.e.*

$$M = \bigcup_{\alpha \in A} U_\alpha.$$

The idea is that if $\phi(p) = (x_1(p), \ldots, x_m(p))$ for $p \in U$ then the functions x_i form a *system of local coordinates* defined on the subset U of M. The *dimension* of M should be m since it takes m numbers to uniquely specify a point of U. We will soon impose conditions on charts (ϕ, U), however *for the moment we are assuming nothing about the maps ϕ* (other than that they are bijective).

Example 1.2.2. Every open subset $U \subset \mathbb{R}^m$ has an atlas consisting of a single chart, namely $(\phi, U) = (\mathrm{id}_U, U)$ where id_U denotes the identity map of U.

Example 1.2.3. Assume that $W \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets, that M is a subset of the product $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$, and $f : W \to V$ is a map whose graph is a subset of M, i.e.

$$\mathrm{graph}(f) := \{(x, y) \in W \times V \mid x \in W, \;\; y = h(x)\} \subset M.$$

Let $U = (W \cap V) \cap \mathrm{graph}(f)$ and let $\phi(x, y) = x$ be the projection of U onto W. Then the pair (ϕ, U) is a chart on M. The inverse map is given by $\phi^{-1}(x) = (x, f(x)))$.

Example 1.2.4. The m-sphere

$$S^m = \{p = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \mid x_0^2 + \cdots + x_m^2 = 1\}$$

[1] See Appendix A.1 for a discussion of the terms injective, surjective, bijective.

has an atlas consisting of the $2m+2$ charts $\phi_{i\pm} : U_{i\pm} \to \mathbb{D}^m$ where $\mathbb{D}^m$ is the open unit disk in $\mathbb{R}^m$, $U_{i\pm} = \{p \in S^m \mid \pm x_i > 0\}$, and $\phi_{i\pm}$ is the projection which discards the ith coordinate. (See Example 2.1.13 below.)

Example 1.2.5. Let $A = A^\mathsf{T} \in \mathbb{R}^{(m+1)\times(m+1)}$ be a symmetric matrix and define a quadratic form $F : \mathbb{R}^{m+1} \to \mathbb{R}$ by

$$F(p) := x^\mathsf{T} A x, \qquad p = (x_0, \ldots, x_m).$$

After a linear change of coordinates the function F has the form

$$F(p) = x_0^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_r^2.$$

(Here r is the rank of the matrix A.) The set $M = F^{-1}(1)$ has an atlas of $2m+1$ charts by the same construction as in Example 1.2.4, in fact S^{m+1} is the special case where $A = \mathbb{1}_n$, the $n \times n$ identity matrix. (See Example 2.1.12 below for another way to construct charts.)

Figure 1.3 enumerates the familiar **quadric surfaces** in $\mathbb{R}^3$. When $W = \mathbb{R}^2$ and $V = \mathbb{R}$ the *paraboloids* are examples of graphs as in Example 1.2.3 and the *ellipsoid* and the two *hyperboloids* are instances of the *quadric surfaces* defined in Example 1.2.5. The sphere is an instance of the ellipsoid ($a = b = c = 1$) and the cylinder is a limit (as $c \to \infty$) of the hyperbolic paraboloid. The pictures were generated by computer using the parameterizations

$$x = a\cos(t)\sin(s), \qquad y = b\sin(t)\sin(s), \qquad z = c\cos(s)$$

for the ellipsoid,

$$x = a\cos(t)\sinh(s), \qquad y = b\sin(t)\sinh(s), \qquad z = c\cosh(s)$$

for the hyperbolic paraboloid, and

$$x = a\cosh(t)\sinh(s), \qquad y = b\sinh(t)\sinh(s), \qquad z = c\cosh(s)$$

for the elliptic paraboloid. These quadric surfaces will be often used in the sequel to illustrate important concepts.

In the following two examples $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, $\mathbb{K}^* := \{\lambda \in K \mid \lambda \neq 0\}$ denote the corresponding multiplicative group, and V denotes a vector space over $\mathbb{K}$.

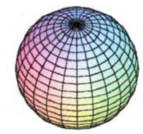
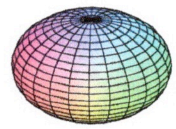
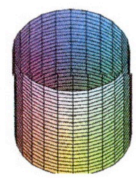
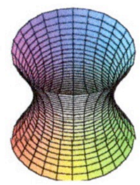
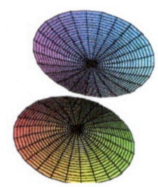
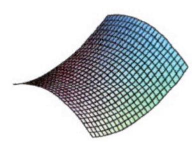
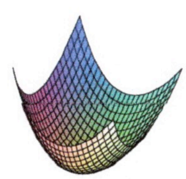
Unit Sphere	$x^2 + y^2 + z^2 = 1$	
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	
Cylinder	$x^2 + y^2 = 1$	
Hyperbolic Hyperboloid *(of one sheet)*	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	
Elliptic Hyperboloid (of two sheets)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	
Hyperbolic Paraboloid	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$	
Elliptic Paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	

Figure 1.3: Quadric Surfaces

Example 1.2.6. The **projective space** of V is the set of lines (through the origin) in V. In other words,

$$P(V) = \{\ell \subset V \mid \ell \text{ is a 1-dimensional } \mathbb{K}\text{-linear subspace}\}$$

When $\mathbb{K} = \mathbb{R}$ and $V = \mathbb{R}^{m+1}$ this is denoted by $\mathbb{RP}^m$ and when $\mathbb{K} = \mathbb{C}$ and $\mathbb{C} = \mathbb{R}^{m+1}$ this is denoted by $\mathbb{CP}^m$. For our purposes we can identify the spaces $\mathbb{C}^{m+1}$ and $\mathbb{R}^{2m+2}$ but the projective spaces $\mathbb{CP}^m$ and $\mathbb{RP}^{2m}$ are very different. The various lines $\ell \in P(V)$ intersect in the origin, however, after the harmless identification

$$P(V) = \{[v] \mid v \in V \setminus \{0\}, \quad [v] := \mathbb{K}^* v = \mathbb{K}v \setminus \{0\}$$

the elements of $P(V)$ become disjoint, i.e. $P(V)$ is the set of equivalence classes of an equivalence relation on the open set $V \setminus \{0\}$. Assume that $V = \mathbb{K}^{m+1}$ and define an atlas on $P(V)$ as follows. For each $i = 0, 1, \ldots, m$ let $U_i = \{[v] \mid, v = (x_0, \ldots, x_m)\ x_i \neq 0\}$ and define a bijection $\phi : U_i \to \mathbb{K}^m$ by the formula

$$\phi_i([v]) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i} \ldots, \frac{x_m}{x_i}\right).$$

This atlas consists of $m + 1$ charts.

Example 1.2.7. For each positive integer k the set

$$G_k(V) := \{\ell \subset V \mid \ell \text{ is a } k\text{-dimensional } \mathbb{K}\text{-linear subspace}\}$$

is called the **Grassmann manifold** of k-planes in V. Thus $G_1(V) = P(V)$. Assume that $V = \mathbb{K}^n$ and define an atlas on $G_k(V)$ as follows. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{K}^n$, i.e. e_i is the ith column of the $n \times n$ identity matrix $\mathbb{1}_n$. Each partition $\{1, 2, \ldots, n\} = I \cup J$, $I = \{i_1 < \cdots < i_k\}$, $J = j_1 < \cdots < j_{n-k}$ of the first the first n natural numbers determines a direct sum decomposition

$$\mathbb{K}^n = V = V_I \oplus V_J$$

via the formulas $V_I = \mathbb{K}e_{i_1} + \cdots + \mathbb{K}e_{i_k}$ and $V_J = \mathbb{K}e_{j_1} + \cdots + \mathbb{K}e_{j_{n-k}}$. Let U_I denote the set of $\ell \in G_k(V)$ which are transverse to V_J, i.e. such that $\ell \cap V_J = \{0\}$. The elements of U_I are precisely those k-planes of form $\ell = \mathrm{graph}(A)$ where $A : V_I \to V_J$ is a linear map. Define $\phi_I : U_i \to \mathbb{K}^{k \times (n-k)}$ by the formula

$$\phi_I(\ell) = (a_{rs}), \qquad Ae_{i_r} = \sum_{s=1}^{n-k} a_{sr} e_{j_s}.$$

Exercise 1.2.8. Prove that the set of all pairs (ϕ_I, U_I) as I ranges over the subsets of $\{1, \ldots, n\}$ of cardinality k form an atlas.