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2.5 Lie Groups

Combining the concept of a group and a manifold, it is interesting to consider groups which are also manifolds and have the property that the group operation and the inverse define smooth maps. We shall only consider groups of matrices.

2.5.1 Definition and Examples

Definition 2.5.1 (Lie Group). *A nonempty subset* $\mathrm{G} \subset \mathbb{R}^{n\times n}$ *is called a* **Lie group** *if it is a submanifold of* $\mathbb{R}^{n\times n}$ *and a subgroup of* $\mathrm{GL}(n,\mathbb{R})$, *i.e.*

$$g, h \in \mathrm{G} \qquad \Longrightarrow \qquad gh \in \mathrm{G}$$

(where gh *denotes the product of the matrices* g *and* h*) and*

$$g \in \mathrm{G} \qquad \Longrightarrow \qquad \det(g) \neq 0 \text{ and } g^{-1} \in \mathrm{G}.$$

(Since $\mathrm{G} \neq \emptyset$ *it follows from these conditions that the identity matrix* $\mathbb{1}$ *is an element of* G.*)*

Example 2.5.2. The general linear group $\mathrm{G} = \mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$ and hence is a Lie group. By Exercise 2.1.18 the special linear group

$$\mathrm{SL}(n,\mathbb{R}) = \{g \in \mathrm{GL}(n,\mathbb{R}) \,|\, \det(g) = 1\}$$

is a Lie group and, by Example 2.1.19, the special orthogonal group

$$\mathrm{SO}(n) := \left\{g \in \mathrm{GL}(n,\mathbb{R}) \,|\, g^{\mathsf{T}} g = \mathbb{1},\, \det(g) = 1\right\}$$

is a Lie group. In fact every orthogonal matrix has determinant ± 1 and so $\mathrm{SO}(n)$ is an open subset of $\mathrm{O}(n)$ (in the relative topology).

In a similar vein the group $\mathrm{GL}(n,\mathbb{C}) := \{g \in \mathbb{C}^{n\times n} \,|\, \det(g) \neq 0\}$ of complex matrices with nonzero (complex) determinant is an open subset of $\mathbb{C}^{n\times n}$ and hence is a Lie group. As in the real case, the subgroups

$$\mathrm{SL}(n,\mathbb{C}) := \{g \in \mathrm{GL}(n,\mathbb{C}) \,|\, \det(g) = 1\},$$

$$\mathrm{U}(n) := \{g \in \mathrm{GL}(n,\mathbb{C}) \,|\, g^* g = \mathbb{1}\},$$

$$\mathrm{SU}(n) := \{g \in \mathrm{GL}(n,\mathbb{C}) \,|\, g^* g = \mathbb{1},\, \det(g) = 1\}$$

are submanifolds of $\mathrm{GL}(n,\mathbb{C})$ and hence are Lie groups. Here $g^* := \bar{g}^{\mathsf{T}}$ denotes the conjugate transpose of a complex matrix.

Exercise 2.5.3. Prove that $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ are Lie groups. Prove that $\mathrm{SO}(n)$ is connected and that $\mathrm{O}(n)$ has two connected components.

Exercise 2.5.4. Prove that $\mathrm{GL}(n,\mathbb{C})$ can be identified with the group

$$\mathrm{G} := \{\Phi \in \mathrm{GL}(2n,\mathbb{R}) \,|\, \Phi J_0 = J_0 \Phi\}, \qquad J_0 := \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Hint: Use the isomorphism $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^n : (x,y) \mapsto x + \mathbf{i}y$. Show that a matrix $\Phi \in \mathbb{R}^{2n\times 2n}$ commutes with J_0 if and only if it has the form

$$\Phi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}, \qquad X, Y \in \mathbb{R}^{n\times n}.$$

What is the relation between the real determinant of Φ and the complex determinant of $X + \mathbf{i}Y$?

Exercise 2.5.5. Let J_0 be as in Exercise 2.5.4 and define

$$\mathrm{Sp}(2n) := \left\{\Psi \in \mathrm{GL}(2n,\mathbb{R}) \,|\, \Psi^{\mathsf{T}} J_0 \Psi = J_0\right\}.$$

This is the **symplectic linear group**. Prove that $\mathrm{Sp}(2n)$ is a Lie group. **Hint:** See [12, Lemma 1.1.12].

Example 2.5.6 (Unit Quaternions). The **Quaternions** form a four-dimensional associative unital algebra $\mathbb{H}$, equipped with a basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$. The elements of $\mathbb{H}$ are vectors of the form

$$x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 \qquad x_0, x_1, x_2, x_3 \in \mathbb{R}. \tag{2.5.1}$$

The product structure is the bilinear map $\mathbb{H} \times \mathbb{H} \to \mathbb{H} : (x,y) \mapsto xy$, determined by the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

This product structure is associative but not commutative. The quaternions are equipped with an involution $\mathbb{H} \to \mathbb{H} : x \mapsto \bar{x}$, which assigns to a quaternion x of the form (2.5.1) its **conjugate** $\bar{x} := x_0 - \mathbf{i}x_1 - \mathbf{j}x_2 - \mathbf{k}x_3$. This involution satisfies the conditions

$$\overline{x+y} = \bar{x} + \bar{y}, \qquad \overline{xy} = \bar{y}\bar{x}, \qquad x\bar{x} = |x|^2, \qquad |xy| = |x|\,|y|$$

for $x, y \in \mathbb{H}$, where $|x| := \sqrt{x_0^2 + x_2^2 + x_2^2 + x_3^2}$ denotes the Euclidean norm of the quaternion (2.5.1). Thus the **unit quaternions** form a group

$$\mathrm{Sp}(1) := \{x \in \mathbb{H} \,|\, |x| = 1\}$$

with the inverse map $x \mapsto \bar{x}$. Note that the group $\mathrm{Sp}(1)$ is diffeomorphic to the 3-sphere $S^3 \subset \mathbb{R}^4$ under the isomorphism $\mathbb{H} \cong \mathbb{R}^4$. **Warning:** The unit quaternions (a compact Lie group) are not to be confused with the symplectic linear group in Exercise 2.5.5 (a noncompact Lie group) despite the similarity in notation.

Let $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie group. Then the maps

$$\mathrm{G} \times \mathrm{G} \to \mathrm{G} : (g, h) \mapsto gh, \qquad \mathrm{G} \to \mathrm{G} : g \mapsto g^{-1}$$

are smooth (see [18]). Fixing an element $h \in \mathrm{G}$ we find that the derivative of the map $\mathrm{G} \to \mathrm{G} : g \mapsto gh$ at $g \in \mathrm{G}$ is given by the linear map

$$T_g\mathrm{G} \to T_{gh}\mathrm{G} : \widehat{g} \mapsto \widehat{g}h. \tag{2.5.2}$$

Here $\widehat{g}$ and h are both matrices in $\mathbb{R}^{n\times n}$ and $\widehat{g}h$ denotes the matrix product. In fact, if $\widehat{g} \in T_g\mathrm{G}$ then, since G is a manifold, there exists a smooth curve $\gamma : \mathbb{R} \to \mathrm{G}$ with $\gamma(0) = g$ and $\dot{\gamma}(0) = \widehat{g}$. Since G is a group we obtain a smooth curve $\beta : \mathbb{R} \to \mathrm{G}$ given by $\beta(t) := \gamma(t)h$. It satisfies $\beta(0) = gh$ and so $\widehat{g}h = \dot{\beta}(0) \in T_{gh}\mathrm{G}$.

The linear map (2.5.2) is obviously a vector space isomorphism whose inverse is given by right multiplication with h^{-1}. It is sometimes convenient to define the map $R_h : \mathrm{G} \to \mathrm{G}$ by

$$R_h(g) := gh$$

for $g \in \mathrm{G}$ (*right multiplication* by h). This is a diffeomorphism and the linear map (2.5.2) is the derivative of R_h at g, so

$$dR_h(g)\widehat{g} = \widehat{g}h \qquad \text{for } \widehat{g} \in T_g\mathrm{G}.$$

Similarly, each element $g \in \mathrm{G}$ determines a diffeomorphism $L_g : \mathrm{G} \to \mathrm{G}$, given by

$$L_g(h) := gh$$

for $h \in \mathrm{G}$ (*left multiplication* by g). Its derivative at $h \in \mathrm{G}$ is again given by matrix multiplication, i.e. the linear map $dL_g(h) : T_h\mathrm{G} \to T_{gh}\mathrm{G}$ is given by

$$dL_g(h)\widehat{h} = g\widehat{h} \qquad \text{for } \widehat{h} \in T_h\mathrm{G}. \tag{2.5.3}$$

Since L_g is a diffeomorphism its differential $dL_g(h) : T_h\mathrm{G} \to T_{gh}\mathrm{G}$ is again a vector space isomorphism for every $h \in \mathrm{G}$.

Exercise 2.5.7. Prove that the map $\mathrm{G} \to \mathrm{G} : g \mapsto g^{-1}$ is a diffeomorphism and that its derivative at $g \in \mathrm{G}$ is the vector space isomorphism

$$T_g\mathrm{G} \to T_{g^{-1}}\mathrm{G} : v \mapsto -g^{-1}vg^{-1}.$$

Hint: Use [18] or any textbook on first year analysis.

2.5.2 The Lie Algebra of a Lie Group

Let

$$\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$$

be a Lie group. Its tangent space at the identity matrix $\mathbb{1} \in \mathrm{G}$ is called the **Lie algebra** of G and will be denoted by

$$\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) := T_{\mathbb{1}}\mathrm{G}.$$

This terminology is justified by the fact that $\mathfrak{g}$ is in fact a Lie algebra, i.e. it is invariant under the standard Lie bracket operation

$$[\xi, \eta] := \xi\eta - \eta\xi$$

on the space $\mathbb{R}^{n\times n}$ of square matrices (see Lemma 2.5.9 below). The proof requires the notion of the **exponential matrix**. For $\xi \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}$ we define

$$\exp(t\xi) := \sum_{k=0}^{\infty} \frac{t^k \xi^k}{k!}. \tag{2.5.4}$$

A standard result in first year analysis asserts that this series converges absolutely (and uniformly on compact t-intervals), that the map

$$\mathbb{R} \to \mathbb{R}^{n\times n} : t \mapsto \exp(t\xi)$$

is smooth and satisfies the differential equation

$$\frac{d}{dt}\exp(t\xi) = \xi \exp(t\xi) = \exp(t\xi)\xi, \tag{2.5.5}$$

and that

$$\exp((s+t)\xi) = \exp(s\xi)\exp(t\xi), \qquad \exp(0\xi) = \mathbb{1} \tag{2.5.6}$$

for all $s, t \in \mathbb{R}$. This shows that the matrix $\exp(t\xi)$ is invertible for each t and that the map $\mathbb{R} \to \mathrm{GL}(n, \mathbb{R}) : t \mapsto \exp(t\xi)$ is a group homomorphism.

Exercise 2.5.8. Prove the following analogue of (2.4.12). For $\xi, \eta \in \mathfrak{g}$

$$\left.\frac{d}{dt}\right|_{t=0} \exp(\sqrt{t}\xi)\exp(\sqrt{t}\eta)\exp(-\sqrt{t}\xi)\exp(-\sqrt{t}\eta) = [\xi, \eta] \tag{2.5.7}$$

In other words, the infinitesimal Lie group commutator is the matrix commutator. (Compare Equations (2.5.7) and (2.4.20).)

Lemma 2.5.9. *Let* $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ *be a Lie group and denote by* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *its Lie algebra. Then the following holds.*

(i) *If* $\xi \in \mathfrak{g}$ *then* $\exp(t\xi) \in \mathrm{G}$ *for every* $t \in \mathbb{R}$.

(ii) *If* $g \in \mathrm{G}$ *and* $\eta \in \mathfrak{g}$ *then* $g\eta g^{-1} \in \mathfrak{g}$.

(iii) *If* $\xi, \eta \in \mathfrak{g}$ *then* $[\xi, \eta] = \xi\eta - \eta\xi \in \mathfrak{g}$.

Proof. We prove (i). For every $g \in \mathrm{G}$ we have a vector space isomorphism $\mathfrak{g} = T_{\mathbb{1}}\mathrm{G} \to T_g\mathrm{G} : \xi \mapsto \xi g$ as in (2.5.2). Hence each element $\xi \in \mathfrak{g}$ determines a vector field $X_\xi \in \mathrm{Vect}(\mathrm{G})$, defined by

$$X_\xi(g) := \xi g \in T_g\mathrm{G}, \qquad g \in \mathrm{G}. \tag{2.5.8}$$

By Theorem 2.4.7 there is an integral curve $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{G}$ satisfying

$$\dot\gamma(t) = X_\xi(\gamma(t)) = \xi\gamma(t), \qquad \gamma(0) = \mathbb{1}.$$

By (2.5.5), the curve $(-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n} : t \mapsto \exp(t\xi)$ satisfies the same initial value problem and hence, by uniqueness, we have $\exp(t\xi) = \gamma(t) \in \mathrm{G}$ for all $t \in \mathbb{R}$ with $|t| < \varepsilon$. Now let $t \in \mathbb{R}$ and choose $N \in \mathbb{N}$ such that $\left|\frac{t}{N}\right| < \varepsilon$. Then $\exp(\frac{t}{N}\xi) \in \mathrm{G}$ and hence it follows from (2.5.6) that

$$\exp(t\xi) = \exp\left(\frac{t}{N}\xi\right)^N \in \mathrm{G}.$$

This proves (i).

We prove (ii). Consider the smooth curve $\gamma : \mathbb{R} \to \mathbb{R}^{n\times n}$ defined by

$$\gamma(t) := g\exp(t\eta)g^{-1}.$$

By (i) we have $\gamma(t) \in \mathrm{G}$ for every $t \in \mathbb{R}$. Since $\gamma(0) = \mathbb{1}$ we have

$$g\eta g^{-1} = \dot\gamma(0) \in \mathfrak{g}.$$

This proves (ii).

We prove (iii). Define the smooth map $\eta : \mathbb{R} \to \mathbb{R}^{n\times n}$ by

$$\eta(t) := \exp(t\xi)\eta\exp(-t\xi).$$

By (i) we have $\exp(t\xi) \in \mathrm{G}$ and, by (ii), we have $\eta(t) \in \mathfrak{g}$ for every $t \in \mathbb{R}$. Hence $[\xi, \eta] = \dot\eta(0) \in \mathfrak{g}$. This proves (iii) and Lemma 2.5.9. □

By Lemma 2.5.9 the curve $\gamma : \mathbb{R} \to \mathrm{G}$ defined by $\gamma(t) := \exp(t\xi)g$ is the integral curve of the vector field X_ξ in (2.5.8) with initial condition $\gamma(0) = g$. Thus X_ξ is complete for every $\xi \in \mathfrak{g}$.

Lemma 2.5.10. *If $\xi \in \mathfrak{g}$ and $\gamma : \mathbb{R} \to \mathrm{G}$ is a smooth curve satisfying*

$$\gamma(s+t) = \gamma(s)\gamma(t), \qquad \gamma(0) = \mathbb{1}, \qquad \dot\gamma(0) = \xi, \tag{2.5.9}$$

then $\gamma(t) = \exp(t\xi)$ for every $t \in \mathbb{R}$.

Proof. For every $t \in \mathbb{R}$ we have

$$\dot\gamma(t) = \left.\frac{d}{ds}\right|_{s=0} \gamma(s+t) = \left.\frac{d}{ds}\right|_{s=0} \gamma(s)\gamma(t) = \dot\gamma(0)\gamma(t) = \xi\gamma(t).$$

Hence γ is the integral curve of the vector field X_ξ in (2.5.8) with $\gamma(0) = \mathbb{1}$. This implies $\gamma(t) = \exp(t\xi)$ for every $t \in \mathbb{R}$, as claimed. □

Example 2.5.11. Since the general linear group $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$ its Lie algebra is the space of all real $n \times n$-matrices

$$\mathfrak{gl}(n, \mathbb{R}) := \mathrm{Lie}(\mathrm{GL}(n, \mathbb{R})) = \mathbb{R}^{n\times n}.$$

The Lie algebra of the special linear group is

$$\mathfrak{sl}(n, \mathbb{R}) := \mathrm{Lie}(\mathrm{SL}(n, \mathbb{R})) = \{\xi \in \mathfrak{gl}(n, \mathbb{R}) \mid \mathrm{trace}(\xi) = 0\}$$

(see Exercise 2.2.8) and the Lie algebra of the special orthogonal group is

$$\mathfrak{so}(n) := \mathrm{Lie}(\mathrm{SO}(n)) = \left\{\xi \in \mathfrak{gl}(n, \mathbb{R}) \mid \xi^\mathsf{T} + \xi = 0\right\} = \mathfrak{o}(n)$$

(see Example 2.2.9).

Exercise 2.5.12. Prove that the Lie algebras of the general linear group over $\mathbb{C}$, the special linear group over $\mathbb{C}$, the unitary group, and the special unitary group are given by

$$\mathfrak{gl}(n, \mathbb{C}) := \mathrm{Lie}(\mathrm{GL}(n, \mathbb{C})) = \mathbb{C}^{n\times n},$$

$$\mathfrak{sl}(n, \mathbb{C}) := \mathrm{Lie}(\mathrm{SL}(n, \mathbb{C})) = \{\xi \in \mathfrak{gl}(n, \mathbb{C}) \mid \mathrm{trace}(\xi) = 0\},$$

$$\mathfrak{u}(n) := \mathrm{Lie}(\mathrm{U}(n)) = \{\xi \in \mathfrak{gl}(n, \mathbb{R}) \mid \xi^* + \xi = 0\},$$

$$\mathfrak{su}(n) := \mathrm{Lie}(\mathrm{SU}(n)) = \{\xi \in \mathfrak{gl}(n, \mathbb{C}) \mid \xi^* + \xi = 0,\ \mathrm{trace}(\xi) = 0\}.$$

These are vector spaces over the reals. Determine their real dimensions. Which of these are also complex vector spaces?

Exercise 2.5.13. Let $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ be a subgroup. Prove that G is a Lie group if and only if it is a closed subset of $\mathrm{GL}(n, \mathbb{R})$ in the relative topology.