diffgeom 57_60, pdf

2.5.3 Lie Group Homomorphisms

Let G and H be Lie groups and $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A **Lie group homomorphism** from G to H is a smooth map $\rho : \mathrm{G} \to \mathrm{H}$ that is a group homomorphism. A **Lie group isomorphism** is a bijective Lie group homomorphism whose inverse is also a Lie group homomorphism. A **Lie algebra homomorphism** from $\mathfrak{g}$ to $\mathfrak{h}$ is a linear map that preserves the Lie bracket.

Lemma 2.5.14. *Let* G *and* H *be Lie groups and denote their Lie algebras by* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *and* $\mathfrak{h} := \mathrm{Lie}(\mathrm{H})$. *Let* $\rho : \mathrm{G} \to \mathrm{H}$ *be a Lie group homomorphism and denote its derivative at* $\mathbb{1} \in \mathrm{G}$ *by*

$$\dot{\rho} := d\rho(\mathbb{1}) : \mathfrak{g} \to \mathfrak{h}.$$

Then $\dot{\rho}$ *is a Lie algebra homomorphism.*

Proof. The proof has three steps.

Step 1. *For all* $\xi \in \mathfrak{g}$ *and* $t \in \mathbb{R}$ *we have* $\rho(\exp(t\xi)) = \exp(t\dot{\rho}(\xi))$.

Fix an element $\xi \in \mathfrak{g}$. Then, by Lemma 2.5.9, we have $\exp(t\xi) \in \mathrm{G}$ for every $t \in \mathbb{R}$. Thus we can define a map $\gamma : \mathbb{R} \to \mathrm{H}$ by $\gamma(t) := \rho(\exp(t\xi))$. Since ρ is smooth, this is a smooth curve in H and, since ρ is a group homomorphism and the exponential map satisfies (2.5.6), our curve γ satisfies the conditions

$$\gamma(s+t) = \gamma(s)\gamma(t), \qquad \gamma(0) = \mathbb{1}, \qquad \dot{\gamma}(0) = d\rho(\mathbb{1})\xi = \dot{\rho}(\xi).$$

Hence it follows from Lemma 2.5.10 that $\gamma(t) = \exp(t\dot{\rho}(\xi))$. This proves Step 1.

Step 2. *For all* $g \in \mathrm{G}$ *and* $\eta \in \mathfrak{g}$ *we have* $\dot{\rho}(g\eta g^{-1}) = \rho(g)\dot{\rho}(\eta)\rho(g^{-1})$.

Define the smooth curve $\gamma : \mathbb{R} \to \mathrm{G}$ by $\gamma(t) := g\exp(t\eta)g^{-1}$. This curve takes values in G by Lemma 2.5.9. By Step 1 we have

$$\rho(\gamma(t)) = \rho(g)\rho(\exp(t\eta))\rho(g)^{-1} = \rho(g)\exp(t\dot{\rho}(\eta))\rho(g)^{-1}$$

for every t. Since $\gamma(0) = \mathbb{1}$ and $\dot{\gamma}(0) = g\eta g^{-1}$ we obtain

$$\begin{aligned} \dot{\rho}(g\eta g^{-1}) &= d\rho(\gamma(0))\dot{\gamma}(0) \\ &= \left.\frac{d}{dt}\right|_{t=0} \rho(\gamma(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} \rho(g)\exp(t\dot{\rho}(\eta))\rho(g^{-1}) \\ &= \rho(g)\dot{\rho}(\eta)\rho(g^{-1}). \end{aligned}$$

This proves Step 2.

Step 3. *For all $\xi, \eta \in \mathfrak{g}$ we have*

$$\dot{\rho}([\xi, \eta]) = [\dot{\rho}(\xi), \dot{\rho}(\eta)].$$

Define the curve $\eta : \mathbb{R} \to \mathfrak{g}$ by

$$\eta(t) := \exp(t\xi)\eta\exp(-t\xi)$$

for $t \in \mathbb{R}$. By Lemma 2.5.9 this curve takes values in the Lie algebra of G and

$$\dot{\eta}(0) = [\xi, \eta].$$

Hence

$$\begin{aligned}
\dot{\rho}([\xi, \eta]) &= \left.\frac{d}{dt}\right|_{t=0} \dot{\rho}\left(\exp(t\xi)\eta\exp(-t\xi)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \rho\left(\exp(t\xi)\right)\dot{\rho}(\eta)\rho\left(\exp(-t\xi)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \exp\left(t\dot{\rho}(\xi)\right)\dot{\rho}(\eta)\exp\left(-t\dot{\rho}(\xi)\right) \\
&= \left[\dot{\rho}(\xi), \dot{\rho}(\eta)\right].
\end{aligned}$$

Here the first equation follows from the fact that $\dot{\rho}$ is linear, the second equation follows from Step 2 with $g = \exp(t\xi)$, and the third equation follows from Step 1. This proves Step 3 and Lemma 2.5.14. □

Example 2.5.15. The complex determinant defines a Lie group homomorphism $\det : \mathrm{U}(n) \to S^1$. The associated Lie algebra homomorphism is

$$\mathrm{trace} = \dot{\det} : \mathfrak{u}(n) \to \mathbf{i}\mathbb{R} = \mathrm{Lie}(S^1).$$

Example 2.5.16 (Unit Quaternions and SU(2)). The Lie group SU(2) is diffeomorphic to the 3-sphere. Every matrix in SU(2) can be written as

$$g = \begin{pmatrix} x_0 + \mathbf{i}x_1 & x_2 + \mathbf{i}x_3 \\ -x_2 + \mathbf{i}x_3 & x_0 - \mathbf{i}x_1 \end{pmatrix}, \qquad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1. \tag{2.5.10}$$

Here the x_i are real numbers. They can be interpreted as the coordinates of a unit quaternion $x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 \in \mathrm{Sp}(1)$ (see Example 2.5.6). The reader may verify that the map $\mathrm{Sp}(1) \to \mathrm{SU}(2) : x \mapsto g$ in (2.5.10) is a Lie group isomorphism.

Exercise 2.5.17 (The double cover of $\mathrm{SO}(3)$**).** Identify the imaginary part of $\mathbb{H}$ with $\mathbb{R}^3$ and write a vector $\xi \in \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ as a purely imaginary quaternion $\xi = \mathbf{i}\xi_1 + \mathbf{j}\xi_1 + \mathbf{k}\xi_3$. Prove that if $\xi \in \mathrm{Im}(\mathbb{H})$ and $x \in \mathrm{Sp}(1)$ then $x\xi\bar{x} \in \mathrm{Im}(\mathbb{H})$. Define the map $\rho : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ by

$$\rho(x)\xi := x\xi\bar{x}$$

for $x \in \mathrm{Sp}(1)$ and $\xi \in \mathrm{Im}(\mathbb{H})$. Prove that the linear map $\rho(x) : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the 3×3-matrix

$$\rho(x) = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_1x_3 + x_0x_2) \\ 2(x_1x_2 + x_0x_3) & x_0^2 + x_2^2 - x_3^2 - x_1^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_2x_3 + x_0x_1) & x_0^2 + x_3^2 - x_1^2 - x_2^2 \end{pmatrix}.$$

Show that ρ is a Lie group homomorphism. Find a formula for the map

$$\dot{\rho} := d\rho(\mathbb{1}) : \mathfrak{sp}(1) \to \mathfrak{so}(3)$$

and show that it is a Lie algebra isomorphism. For $x, y \in \mathrm{Sp}(1)$ prove that $\rho(x) = \rho(y)$ if and only if $y = \pm x$.

Example 2.5.18. Consider the map

$$\mathrm{GL}(n, \mathbb{R}) \to \mathrm{Diff}(\mathbb{R}^n) : g \mapsto \phi_g$$

which assigns to every nonsingular matrix $g \in \mathrm{GL}(n, \mathbb{R})$ the linear diffeomorphism $\phi_g : \mathbb{R}^n \to \mathbb{R}^n$ given by $\phi_g(x) := gx$ for $x \in \mathbb{R}^n$. This map $g \mapsto \phi_g$ is a group homomorphism. The group $\mathrm{Diff}(\mathbb{R}^n)$ is infinite dimensional and thus cannot be a Lie group. However, it has many properties in common with Lie groups. For example one can define what is meant by a smooth path in $\mathrm{Diff}(\mathbb{R}^n)$ and extend formally the notion of a tangent vector (as the derivative of a path through a given element of $\mathrm{Diff}(\mathbb{R}^n)$) to this setting. In particular, the tangent space of $\mathrm{Diff}(\mathbb{R}^n)$ at the identity can then be identified with the space of vector fields

$$T_{\mathrm{id}}\mathrm{Diff}(\mathbb{R}^n) = \mathrm{Vect}(\mathbb{R}^n).$$

Differentiating the map $g \mapsto \phi_g$, one then obtains a linear map

$$\mathfrak{gl}(n, \mathbb{R}) \to \mathrm{Vect}(\mathbb{R}^n) : \xi \mapsto X_\xi$$

which assigns to every matrix $\xi \in \mathfrak{gl}(n, \mathbb{R})$ the vector field $X_\xi : \mathbb{R}^n \to \mathbb{R}^n$ given by $X_\xi(x) := \xi x$ for $x \in \mathbb{R}^n$. We have already seen in Remark 2.4.23 that this map is a Lie algebra homomorphism.

Example 2.5.19. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then the set

$$\mathrm{Aut}(\mathfrak{g}) := \left\{ \Phi : \mathfrak{g} \to \mathfrak{g} \;\middle|\; \begin{array}{l} \Phi \text{ is a bijective linear map,} \\ \Phi[\xi, \eta] = [\Phi\xi, \Phi\eta] \;\forall\, \xi, \eta \in \mathfrak{g} \end{array} \right\}$$

of **Lie algebra automorphisms** of $\mathfrak{g}$ is a Lie group. Its Lie algebra is the space of **derivations** on $\mathfrak{g}$ denoted by

$$\mathrm{Der}(\mathfrak{g}) := \left\{ A : \mathfrak{g} \to \mathfrak{g} \;\middle|\; \begin{array}{l} A \text{ is a linear map,} \\ A[\xi, \eta] = [A\xi, \eta] + [\xi, A\eta] \;\forall\, \xi, \eta \in \mathfrak{g} \end{array} \right\}.$$

Now suppose that $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ is the Lie algebra of a Lie group G. Then there is a map

$$\mathrm{ad} : \mathrm{G} \to \mathrm{Aut}(\mathfrak{g}), \qquad \mathrm{ad}(g)\eta := g\eta g^{-1}, \tag{2.5.11}$$

for $g \in \mathrm{G}$ and $\eta \in \mathfrak{g}$. Lemma 2.5.9 (ii) asserts that $\mathrm{ad}(g)$ is indeed a linear map from $\mathfrak{g}$ to itself for every $g \in \mathrm{G}$. The reader may verify that the map

$$\mathrm{ad}(g) : \mathfrak{g} \to \mathfrak{g}$$

is a Lie algebra automorphism for every $g \in \mathrm{G}$ and that the map $\mathrm{ad} : \mathrm{G} \to \mathrm{Aut}(\mathfrak{g})$ is a Lie group homomorphism. The associated Lie algebra homomorphism is the map

$$\mathrm{Ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}), \qquad \mathrm{Ad}(\xi)\eta := [\xi, \eta], \tag{2.5.12}$$

for $\xi, \eta \in \mathfrak{g}$. To verify the claim $\mathrm{Ad} = \dot{\mathrm{ad}}$ we compute

$$\dot{\mathrm{ad}}(\xi)\eta = \left.\frac{d}{dt}\right|_{t=0} \mathrm{ad}(\exp(t\xi))\eta = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)\eta\exp(-t\xi) = [\xi, \eta].$$

Exercise 2.5.20. Let $\mathfrak{g}$ be any Lie algebra and define the map

$$\mathrm{Ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

by (2.5.12). Prove that the endomorphism

$$\mathrm{Ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$$

is a derivation for every $\xi \in \mathfrak{g}$ and that $\mathrm{Ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is a Lie algebra homomorphism. If $\mathfrak{g}$ is finite dimensional, prove that $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathrm{Der}(\mathfrak{g})$.