diff geom G_64

2.5.4 Lie Groups and Diffeomorphisms

There is a natural correspondence between Lie groups and Lie algebras on the one hand and diffeomorphisms and vector fields on the other hand. We summarize this correspondence in the following table

Lie groups	**Diffeomorphisms**
$\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$	$\mathrm{Diff}(M)$
$\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}$	$\mathrm{Vect}(M) = T_{\mathrm{id}}\mathrm{Diff}(M)$
exponential map	flow of a vector field
$t \mapsto \exp(t\xi)$	$t \mapsto \phi^t =$ "$\exp(tX)$"
adjoint representation	pushforward
$\xi \mapsto g\xi g^{-1}$	$X \mapsto \phi_* X$
Lie bracket on $\mathfrak{g}$	Lie bracket of vector fields
$[\xi, \eta] = \xi\eta - \eta\xi$	$[X, Y] = dX \cdot Y - dY \cdot X$.

To understand the correspondence between the exponential map and the flow of a vector field compare equation (2.4.6) with equation (2.5.5). To understand the correspondence between the adjoint representation and pushforward observe that

$$\phi_* Y = \left.\frac{d}{dt}\right|_{t=0} \phi \circ \psi^t \circ \phi^{-1}, \qquad g\eta g^{-1} = \left.\frac{d}{dt}\right|_{t=0} g \exp(t\eta) g^{-1},$$

where ψ^t denotes the flow of Y. To understand the correspondence between the Lie brackets recall that

$$[X, Y] = \left.\frac{d}{dt}\right|_{t=0} (\phi^t)_* Y, \qquad [\xi, \eta] = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)\eta\exp(-t\xi),$$

where ϕ^t denotes the flow of X. We emphasize that the analogy between Lie groups and Diffeomorphisms only works well when the manifold M is compact so that every vector field on M is complete. The next exercise gives another parallel between the Lie bracket on the Lie algebra of a Lie group and the Lie bracket of two vector fields.

Exercise 2.5.21. Let $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $\xi, \eta \in \mathfrak{g}$. Define the smooth curve $\gamma : \mathbb{R} \to \mathrm{G}$ by

$$\gamma(t) := \exp(t\xi)\exp(t\eta)\exp(-t\xi)\exp(-t\eta).$$

Prove that $\dot{\gamma}(0) = 0$ and $\frac{1}{2}\ddot{\gamma}(0) = [\xi, \eta]$. Compare this with Lemma 2.4.18.

Exercise 2.5.22. Let $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $\xi, \eta \in \mathfrak{g}$. Show that $[\xi, \eta] = 0$ if and only if the exponential maps commute, i.e. $\exp(s\xi)\exp(t\eta) = \exp(t\eta)\exp(s\xi)$ for all $s, t \in \mathbb{R}$. How can this observation be deduced from Lemma 2.4.26?

2.5.5 Smooth Maps and Algebra Homomorphisms

Let M be a smooth submanifold of $\mathbb{R}^k$. Denote by $\mathscr{F}(M) := C^\infty(M, \mathbb{R})$ the space of smooth real valued functions $f : M \to \mathbb{R}$. Then $\mathscr{F}(M)$ is a commutative unital algebra. Each $p \in M$ determines a unital algebra homomorphism $\varepsilon_p : \mathscr{F}(M) \to \mathbb{R}$ defined by $\varepsilon_p(f) = f(p)$ for $p \in M$.

Theorem 2.5.23. *Every unital algebra homomorphism $\varepsilon : \mathscr{F}(M) \to \mathbb{R}$ has the form $\varepsilon = \varepsilon_p$ for some $p \in M$.*

Proof. Assume that $\varepsilon : \mathscr{F}(M) \to \mathbb{R}$ is an algebra homomorphism.
Claim. *For all $f, g \in \mathscr{F}(M)$ we have $\varepsilon(g) = 0 \implies \varepsilon(f) \in f(g^{-1}(0))$.*
Indeed, the function $f - \varepsilon(f) \cdot 1$ lies in the kernel of ε and so the function $h := (f - \varepsilon(f) \cdot 1)^2 + g^2$ also lies in the kernel of ε. There must be at least one point $p \in M$ where $h(p) = 0$ for otherwise $1 = \varepsilon(h)\varepsilon(1/h) = 0$. For this point p we have $f(p) = \varepsilon(p)$ and $g(p) = 0$, hence $p \in g^{-1}(0)$, and therefore $\varepsilon(f) = f(p) \in f(g^{-1}(0))$. This proves the claim.

The theorem asserts that there exists a $p \in M$ such that every $f \in \mathscr{F}(M)$ satisfies $\varepsilon(f) = f(p)$. Assume, by contradiction, that this is false. Then for every $p \in M$ there exists a function $f \in \mathscr{F}(M)$ such that $f(p) \neq \varepsilon(f)$. Replace f by $f - \varepsilon(f)$ to obtain $f(p) \neq 0 = \varepsilon(f)$. Now use the axiom of choice to obtain a family of functions $f_p \in \mathscr{F}(M)$, one for every $p \in M$, such that $f_p(p) \neq 0 = \varepsilon(f_p)$ for all $p \in M$. Then the set $U_p := f_p^{-1}(\mathbb{R} \setminus \{0\})$ is an M-open neighborhood of p for every $p \in M$. Choose a sequence of compact sets $K_n \subset M$ such that $K_n \subset \mathrm{int}_M(K_{n+1})$ for all n and $M = \bigcup_n K_n$. Then, for each n, there is a $g_n \in \mathscr{F}(M)$ (a finite sum of the form $\sum_i f_{p_i}^2$) such that $\varepsilon(g_n) = 0$ and $g_n(q) > 0$ for all $q \in K_n$. If M is compact, this is already a contradiction because a positive function cannot belong to the kernel of ε. Otherwise, choose $f \in \mathscr{F}(M)$ such that $f(q) \geq n$ for all $q \in M \setminus K_n$ and all $n \in \mathbb{N}$. Then $\varepsilon(f) \in f(g_n^{-1}(0)) \subset f(M \setminus K_n) \subset [n, \infty)$ by the claim and so $\varepsilon(f) \geq n$ for all n. This is a contradiction and proves Theorem 2.5.23. □

Now let N be another smooth submanifold (say of $\mathbb{R}^\ell$) and let $C^\infty(M, N)$ denote the space of smooth maps from M to N. A **homomorphism** from $\mathscr{F}(N)$ to $\mathscr{F}(M)$ is a (real) linear map $\Phi : \mathscr{F}(N) \to \mathscr{F}(M)$ that satisfies

$$\Phi(fg) = \Phi(f)\Phi(g), \qquad \Phi(1) = 1.$$

An **automorphism** of the algebra $\mathscr{F}(M)$ is a bijective homomorphism $\Phi : \mathscr{F}(M) \to \mathscr{F}(M)$. Let $\mathrm{Hom}(\mathscr{F}(N), \mathscr{F}(M))$ denote the space of homomorphisms from $\mathscr{F}(N)$ to $\mathscr{F}(M)$. The automorphisms of $\mathscr{F}(M)$ form a group denoted by $\mathrm{Aut}(\mathscr{F}(M))$.

Corollary 2.5.24. *The pullback operation*

$$C^\infty(M,N) \to \mathrm{Hom}(\mathscr{F}(N),\mathscr{F}(M)) : \phi \mapsto \phi^*$$

is bijective. In particular, the map $\mathrm{Diff}(M) \to \mathrm{Aut}(\mathscr{F}(M)) : \phi \mapsto \phi^*$ *is an anti-isomorphism of groups.*

Proof. This is an exercise with hint. Let $\Phi : \mathscr{F}(N) \to \mathscr{F}(M)$ be a unital algebra homomorphism. By Theorem 2.5.23 there exists a map $\phi : M \to N$ such that $\varepsilon_p \circ \Phi = \varepsilon_{\phi(p)}$ for all $p \in M$. Prove that $f \circ \phi : M \to \mathbb{R}$ is smooth for every smooth map $f : N \to \mathbb{R}$ and deduce that ϕ is smooth. □

Remark 2.5.25. The pullback operation is **functorial**, i.e.

$$(\psi \circ \phi)^* = \psi^* \circ \phi^*, \qquad \mathrm{id}_M^* = \mathrm{id}_{\mathscr{F}(M)}.$$

for $\phi \in C^\infty(M,N)$ and $\psi \in C^\infty(N,P)$. Here id denotes the identity map of the space indicated in the subscript. Hence Corollary 2.5.24 may be summarized by saying that the category of smooth manifolds and smooth maps is anti-isomorphic to a subcategory of the category of commutative unital algebras and unital algebra homomorphisms.

Exercise 2.5.26. If M is compact, then there is a slightly different way to prove Theorem 2.5.23. An **ideal** in $\mathscr{F}(M)$ is a linear subspace $\mathscr{J} \subset \mathscr{F}(M)$ satisfying the condition $f \in \mathscr{F}(M),\ g \in \mathscr{J} \implies fg \in \mathscr{J}$. A **maximal ideal** in $\mathscr{F}(M)$ is an ideal $\mathscr{J} \subsetneq \mathscr{F}(M)$ such that every ideal $\mathscr{J}' \subsetneq \mathscr{F}(M)$ containing $\mathscr{J}$ is equal to $\mathscr{J}$. Prove that, if M is compact and $\mathscr{J} \subset \mathscr{F}(M)$ is an ideal with the property that for every $p \in M$ there is an $f \in \mathscr{J}$ with $f(p) \neq 0$, then $\mathscr{J} = \mathscr{F}(M)$. Deduce that each maximal ideal in $\mathscr{F}(M)$ has the form $\mathscr{J}_p := \{f \in \mathscr{F}(M) \mid f(p) = 0\}$ for some $p \in M$.

Exercise 2.5.27. If M is compact, give another proof of Corollary 2.5.24 as follows. The set $\Phi^{-1}(\mathscr{J}_p)$ is a maximal ideal in $\mathscr{F}(N)$ for each $p \in M$. Use Exercise 2.5.26 to deduce that there is a unique map $\phi : M \to N$ such that $\Phi^{-1}(\mathscr{J}_p) = \mathscr{J}_{\phi(p)}$ for all $p \in M$. Show that ϕ is smooth and $\phi^* = \Phi$.

Exercise 2.5.28. It is a theorem of ring theory that, when $I \subset R$ is an ideal in a ring R, the quotient ring R/I is a field if and only if the ideal I is maximal. Show that the kernel of the ring homomorphism $\varepsilon_p : \mathscr{F}(M) \to \mathbb{R}$ of Theorem 2.5.23 is the ideal $\mathscr{J}_p$ of Exercise 2.5.26. Conclude that M is compact if and only if every maximal ideal $\mathscr{J}$ in $\mathscr{F}(M)$ is of the form $\mathscr{J} = \mathscr{J}_p$ for some $p \in M$. **Hint:** The functions of compact support form an ideal. It can be shown that if M is not compact and $\mathscr{J}$ is a maximal ideal containing all functions of compact support then the quotient field $\mathscr{F}(M)/\mathscr{J}$ is a non-Archimedean ordered field which properly contains $\mathbb{R}$.