Need $\|a^*\| = \|a\|$

$\|a^*a\| = \|a\|^2$

part 4

11-16-17 II

(1)

unitaries are extreme points & conversely

extreme pts. pdf

$$u = \frac{a+b}{2}$$

$$uu^* = \frac{a+b}{2} \cdot \frac{a^*+b^*}{2} = \frac{a^*+b^*}{2} \cdot \frac{a+b}{2}$$

$$1 = \frac{aa^* + ba^* + ab^* + bb^*}{2} = \frac{a^*a + b^*a + a^*b + b^*b}{2}$$

$$1 = aa^* = bb^* = ab^* = ba^* = a^*a = b^*a = a^*b = b^*b$$

$$b^* = a^*ab^* = a^*ba^* = a^*$$

$$b = a$$

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need $1 = \frac{a+b}{2}$ $\quad \|a\| = \|b\| = 1 \Rightarrow 1 = a = b$

$\|c\| \le 1$

let $c = \frac{a+a^*}{2}$ $\quad d = \frac{b+b^*}{2}$ $\quad \|d\| \le 1$

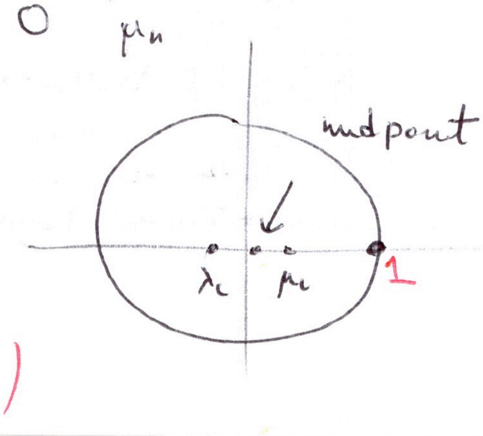
$c^* = c \quad d^* = d$

$1 = \frac{c+d}{2}$ $\quad d = 2 - c$ $\quad$ so $dc = cd$

eigenvalues $\quad$ eigenvalues real

simultaneously diagonal $\quad wdw^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ $\quad wcw^{-1} = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix}$ $\quad |\lambda_i| \le 1$, $|\mu_i| \le 1$

$$\frac{\lambda_i + \mu_i}{2} = 1 \qquad \lambda_i = \mu_i$$



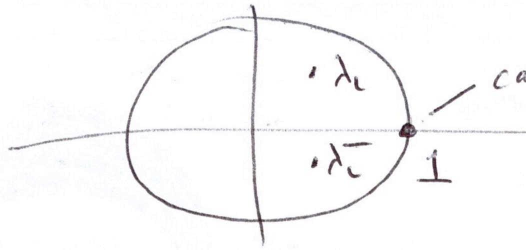
$$1 = w1w^{-1} = \frac{wcw^{-1} + wdw^{-1}}{2}$$

$a + a^* = 2$ $\quad$ a is normal $\quad (b + b^* = 2)$

$$1 = \frac{a + a^*}{2} \qquad a = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad \lambda_i \text{ complex}$$

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1 + \bar{\lambda}_1}{2} & & 0 \\ & \ddots & \\ 0 & & \frac{\lambda_n + \bar{\lambda}_n}{2} \end{pmatrix} \qquad |\lambda_i| \leq 1$$

$$\lambda_i + \bar{\lambda}_i = 2$$



cannot be midpoint of $\lambda_i, \bar{\lambda}_i$

unitaries are extreme points

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converse let x be extreme point $\quad x = u|x|$ polar decomp

mult by u^{-1} is an isometry $\qquad |x| = (x^* x)^{1/2}$

so $|x|$ is an extreme point $\qquad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n = 1$

$$|x| = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad \lambda_i \geq 0 \qquad \max \lambda_i = 1$$

claim $\lambda_i = 1 \ \forall i$

say $0 < \lambda_1 < 1$

$$|x| = \lambda_1 \begin{pmatrix} 1 & & \\ & \lambda_2 & 0 \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix} + (1 - \lambda_1) \begin{pmatrix} 0 \\ \\ \\ \end{pmatrix}$$

Now to prove extreme pts

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Sakai $x \in \mathrm{ext}$ x^*x

claim: x^*x is a projection

$$x^*x = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad 1 \geq \lambda_i \geq 0$$

$x^*x(1+c)^2$

$$= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} (1+c_1)^2 & & 0 \\ & 1 & \\ 0 & & \ddots \end{pmatrix}$$

$$\begin{matrix} \lambda_1(1+c_1)^2 & & \\ & \lambda_2 & \\ & & \ddots \;\lambda_n \end{matrix}$$

say $\lambda_1 < 1$

$$c = \begin{pmatrix} c_1 \neq 0 & & \\ & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} > 0$$

$1+c$ $\quad 1-c$

$$\| x^*x(1+c)^2 \| \leq 1$$

$$\| x^*x(1-c)^2 \| \leq 1$$

So $x(1+c)$, $x(1-c)$ have norm ≤ 1

$$x = \frac{x+xc + x-xc}{2}$$

$\lambda_1(1+c_1)^2 \leq 1$

$\lambda_1 \leq (1+c_1)^2$

$$x = x+xc = x-xc$$

$xc = 0$, $0 = \| c x^*x c \| = \| x^*x c^2 \| \stackrel{?}{=} 0$

contradiction since $x^*x c^2 \neq 0$

$$1+c_1 \leq \frac{1}{\sqrt{\lambda_1}}$$

$$c_1 \leq \frac{1}{\sqrt{\lambda_1}} - 1 \quad \underline{\underline{\text{can do}}} \text{ since } \lambda_1 < 1, \ \sqrt{\lambda_1} < 1$$

$x^*x(1+c)^2$

$$= \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} (1+c_1)^2 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1(1+c_1)^2 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

need $\lambda_1(1+c_1)^2 \leq 1$

$(1+c_1)^2 \leq \frac{1}{\lambda_1}$

$$pAq = 0 \overset{?}{\Longrightarrow} p = q = 0$$

$pq = 0$ $p = \begin{matrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & 0 \end{matrix}$ $q = \begin{matrix} 0 & & 0 \\ & 0 & \\ 0 & & 1 \\ & & & 1 \end{matrix}$

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \vdots \\ a_{n1} & & a_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_1 a_{11} & \lambda_1 a_{12} & \cdots & \lambda_1 a_{1n} \\ \lambda_2 a_{21} & \lambda_2 a_{22} & \cdots & \lambda_2 a_{2n} \\ \vdots & & & \\ \lambda_n a_{n1} & \cdots & & \lambda_n a_{nn} \end{pmatrix}$$

$$0 = paq = \begin{array}{c|c|c|c} & & q & \\ p & 0 \; \cancel{0\,0\,0} & & 0 \\ & 0 \; \cancel{0\,0\,0} & & 0 \\ & & 0 \; 0 & * \\ & * & a_{ij} \; 0 & \\ & * & 0 \; 0 & * \end{array}$$

false if $a_{ij} \neq 0$

$$aq = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & & \\ 0 & 0 & \lambda_3 & \\ & & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 a_{11} & \lambda_2 a_{12} & \cdots & \lambda_n a_{1n} \\ \lambda_1 a_{21} & \lambda_2 a_{22} & \cdots & \lambda_n a_{2n} \\ \vdots & & & \\ \lambda_1 a_{n1} & \cdots & & \lambda_n a_{nn} \end{pmatrix}$$

$A = pAq + (1-p)Aq + pA(1-q) + (1-p)A(1-q)$

$1-p$, p, p, 0

$A = pAp + \cancel{pA(1-p)} + (1-p)Ap + (1-p)A(1-p)$

0

$i > 2$

$j < 3$

contradiction

$$A = \begin{pmatrix} pAp & 0 \\ 0 & + (1-p)A(1-p) \end{pmatrix}$$

$p + q = 1$

$1 - x^* x + 1 - xx^* = 1$

$1 - x^* x + xx^*$

$$\boxed{1 = x^* x + xx^*}$$

$p + q$

$x^* x = 1 - xx^*$

$xx^* A (1 - x^* x)$