LOVE AND MATH
Unlocking the power and beauty of mathematics

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Chapter 1. A Mysterious Beast

What really excited me was physics—especially quantum physics

Atoms were proved to exist at the beginning of the 20th century. At the same time, scientists discovered that each atom could be further divided

I thought it was really cool that a physicist would name a particle after a novel (Murray Gell-Mann, quarks)

All of these (other) interests made me doubt whether I was really cut out to be a scientist.

Neutrons and protons are both hadrons, Gell-Mann and independently, Yuval Ne'eman proposed a novel classification scheme. They both showed that hadrons can be naturally split into small families, each consisting of 8 or 10 particles.

I must meet him. I will try to convert him to math (Evgeny Evgenievich)

So, I hear that you like quantum physics. How can you possibly understand the quark model if you don't know what the group $SU(3)$ is?

You probably thought that mathematics is what they teach you in school. No, this is what mathematics is about

The first thing you need to learn is the concept of a **symmetry group**.

Evgeny Evgenievich found a perfect combination of topics that would allow me to see this mysterious beast —*Mathematics*— from different sides and get excited about it

At school, we studied things like quadratic equations, a bit of calculus, some basic Euclidean geometry, and trigonometry. The books Evgeny Evgenievich gave me contained glimpses of an entirely different world, whose existence I couldn't even imagine. I WAS INSTANTLY CONVERTED

Chapter 2. The Essence of Symmetry

In the minds of most people, mathematics is all about numbers.

What is symmetry? A square table has four symmetries. A round table has many more symmetries (infinitely many)

The **identity** symmetry; **composition** of symmetries; the **inverse** of a symmetry. These are the basic ingredients of what mathematicians call a **group**

Examples: the group of symmetries of a square table, the group of symmetries of a round table

Experience shows that symmetry is an essential guiding principle for the laws of nature

There are many objects in nature whose symmetries are approximate. It is useful then to consider their abstract, idealized version, or models.

The main point of the mathematical theory of symmetry is not aesthetic. It is to formulate the concept of symmetry in the most general, and hence inevitably most abstract, terms, so that it can be applied in a unified fashion in different domains, such as geometry, number theory, physics, chemistry, biology, and so on.

The basic qualities of the abstract theory of symmetry (or, why mathematics is important): **universality, objectivity, endurance, relevance** (to the physical world)

Perfect example: the discovery of quarks, based on representations of the symmetry group $SU(3)$.

The term **representation**, as used in mathematics

If each element of a group can be realized, in a consistent manner, as a symmetry of an n-dimensional space, then we say that the group has an n-dimensional representation.

$SU(3)$ is known to have an 8-dimensional and a 10-dimenstional representation. Mathematicians have proved that $SU(3)$ has no 7-dimensional or 11-dimensional representations.

The mathematical theory of representations of the group $SU(3)$ motivated Gell-Mann and Zweig to predict the existence of quarks. These particles were not predicted on the basis of empirical data, but on the basis of mathematical symmetry patterns.

It took physicists years to master this theory (and in fact there was some resistance to it at first), but it is now the bread and butter of elementary particle physics. Not only did it provide a classification of hadrons, it also led to the discovery of quarks.

Physicists do need expensive and sophisticated machines such as the Large Hadron Collider in Geneva, but the amazing fact is that scientists like Einstein and Gell-Mann have used what looks like the purest and most abstract mathematical knowledge to unlock the deepest secrets of the world around us

Chapter 3. The Fifth Problem

This is what happens when you fall in love

Moscow has many schools, but there was only one place to study pure math: Moscow State University

In the Soviet Union circa 1984—remember Orwell?—it was not considered bizarre to ask someone what his or her "nationality" was. This was a code for asking whether one was Jewish or not

Do you know that Jews are not accepted to Moscow University? The government didn't want Jews in a program related to nuclear research and hence to national defense and state secrets because they could emigrate to Israel or elsewhere

The first exam was a written test in mathematics consisting of five problems, the fifth being deadly and unsolvable.

The next exam was oral math. I was ready. I raised my hand. They ignored me, as if I didn't exist. Every word I said was questioned.

Mark Saul, *Kerosinka; An episode in the history of Soviet mathematics*, Notices of the American Mathematical Society, vol. 46, November 1999, pp. 1217–1220. Available online at http://www.ams.org/notices/199910/fea-saul.pdf

The Red Queen interrogating Alice in *Alice in Wonderland*

George G. Szpiro, *Bella Abramovna Subbotovskaya and the "Jewish People's University,"* Notices of the American Mathematical Society, vol. 54, November 2007, pp. 1326–1330. Available online at http://www.ams.org/notices/200710/tx071001326p.pdf

Go to the Moscow Institute of Oil and Gas. They have an applied mathematics program, which is quite good. They take students *like you* there.

Chapter 4. Kerosinka

Like millions of people, my grandfather had been a victim of Joseph Stalin's persecution. In 1948, he was sent to a hard-labor camp at a coal mine in northern Russia, part of the Gulag Archipelago that Alexander Solzhenitsyn and other writers described so vividly years later.

My father was therefore a "son of the enemy of the people" hence in 1954 was denied admission to the physics department of Gorky University.

By a general decree of Nikita Khrushchev in 1956, my grandfather was released from the hard-labor camp. But by then it was too late to undo the injustice done to my father. Now, 30 years later, his son had to go through a similar experience

From the late 1960s, anti-Semitism at Moscow University created a market for placements in mathematics for Jewish students, including at Moscow Institute of Oil and Gas. (Mark Saul article)

Being exposed to applied mathematics courses taught me that there isn't really a sharp distinction between "pure" and "applied" math; good-quality applied math is always based on sophisticated pure math

I had to find a way to learn the pure math subjects that we not offered at Kerosinka. ⋯ That sounded dangerous and exciting, so I said "Sure."

In the meantime, I was learning all the math I could at Kerosinka and meeting with Evgeny Evgenievich every couple of weeks

To maintain my momentum, as well as the motivation for it, I would need an advisor with whom I could meet more regularly and not only learn from, but also get a problem to work on.

With all odds stacked against me, I started to doubt that I could fulfill my dream of becoming a mathematician.

Chapter 5. Threads of the Solution

Would you be interested in working on a math problem?

If Varchenko's problem had been closely tied to his own research program, he might have tried to solve it himself. But no mathematician does everything alone. This was a typical "transaction" in the social workings of the mathematical world

Without Fuchs' kindness and generosity, I would never have become a mathematician. Having an advisor is absolutely essential.

Try to read this, and as soon as you see a word that you don't understand, call me.

I had heard of the "braid groups" before. These are excellent examples of groups, the concept we discussed in chapter 2.

Once we have the notion "group," we can look for other examples. It turns out there are many examples of groups that have nothing to do with symmetries.

This is a typical story. The creation of a mathematical concept may be motivated by problems and phenomena in one area of math (or physics, engineering, etc) but later it may well turn out to be useful and well adapted to other areas

I did not know yet the real-world applications of braid groups to such areas as cryptography, quantum computing, and biology, which we will talk about later

There is one braid group, denoted by B_n, for each natural number $n = 1, 2, \ldots$.

The elements of B_n are the so-called braids with n threads, where each thread is allowed to weave around any other thread any way we like.

These are some "hairy numbers," if you will

What is the composition of two braids with n threads? What is the inverse of a braid? What is the identity braid? Is B_n a group?

We do not distinguish two braids which can be obtained from one another by pulling the threads, or by stretching or shrinking them any way we like so long as we do not cut or resew the threads

B_1 is the trivial one-element group consisting of the identity. B_2 has the same structure as the group of integers, so it is "commutative." B_n for $n = 3, 4, \ldots$ is "non-commutative."

See endnotes 7,8.9 of this chapter for references for the applications of braid groups mentioned above (Not for everyone)

In mathematics, braids are also important because of their geometric interpretation. See endnote 10 of this chapter for more details on this point.

See endnotes 11,12,13 of chapter 5 for details of the problem Fuchs gave to our hero. (Not for everyone; neither is endnote 10 which refers to chapters 9 and 14)

Chapter 6. Apprentice Mathematician

Solving a mathematical problem is like doing a jigsaw puzzle, except you don't know in advance what the final picture will look like

It could be hard, it could be easy, or it could be impossible. You never know until you actually do it (or realize that it's impossible to do)

This uncertainty is perhaps the most difficult aspect of being a mathematician

In math, the problem is always well defined, and there is no ambiguity about what solving it means.

The problem I solved means the same thing today to everyone familiar with the language of math, as it did in 1986, and will mean the same thing a hundred years from now

Fermat's Last Theorem

$x + y = z$; $x^2 + y^2 = z^2$ YES $x^3 + y^3 = z^3$; $x^4 + y^4 = z^4 \ldots$ NO

In 1637, Fermat wrote that he had found a simple proof of this statement, for all n greater than 2, but "this margin is too small to contain it."

In 1993, a Princeton mathematician, Andrew Wiles, announced his proof of Fermat's Last Theorem.

It is clear to us now that Fermat could not have possibly proved the statement attributed to him.

Entire fields of mathematics had to be created in order to do this, a development that took a lot of hard work by many generations of mathematicians.

Despite my classes and exams at Kerosinka, my highest priority was my problem, and I spent endless hours with it, nights and weekends

A side effect of this is that I had trouble sleeping. After that I never allowed myself to get lost so completely in a math problem involved

For the first time in my life, I had in my possession something that no one in the world had. It wasn't a cure for cancer, but it was a worthy piece of knowledge

The problem I solved suggested the existence of some hidden connections between braid groups and number theory. Thus it could have potential implications far beyond its original scope

Searching for new patterns on the edge of knowledge was captivating and exciting. Sitting at my desk, trying to organize my thoughts and put them on paper, was an entirely different process

Israel Moiseevich Gelfand

Patriarch of the Soviet mathematical school. Sole editor of the most influential Soviet math journal. Presided over a weekly seminar renowned all over the world

The seminar was in many ways the theater of a single actor (dictator!). One of his skills was the ability to "rephrase" questions asked by others in such a way that the answer became obvious

A seminar like this could only exist in a totalitarian society. Whatever one can say about Gelfand's style, people never left the seminar empty-handed

"Interesting, but why is this important?" This has to be written clearly in the paper. Otherwise the paper looks good to me

Chapter 7. The Grand Unified Theory

The solution of the first problem was my initiation into the temple of mathematics. . . . , but my goal in this book is to describe much more than my own experience

It is to give you a sense of modern math, to prove that it is really about originality, imagination, groundbreaking insights. The Langlands Program is a great example

Mathematics consists of many subfields. They often feel like different continents, with mathematicians working in those subfields speaking different languages

That's why the idea of "unification," bringing together the theories coming from these diverse fields and realizing that they are all part of a single narrative, is so powerful

Every once in a while, someone will come who will see how to connect various subfields. This is what Robert Langlands did in the late 1960s

The key point of the Langlands Program is the concept of symmetry. The symmetry groups that are relevant here appear in the theory of numbers

The vast majority of the numbers that we encounter in everyday life are fractions, or rational numbers. But there are also numbers which are not rational

Since $\sqrt{2}$ is the length of the hypotenuse of a certain right triangle, we know that this number is out there. (It is also one of the solutions to the equation $x^2 = 2$) But it just does not fit in the numerical system of rational numbers

Let's drop $\sqrt{2}$ in the rationals and see what kind of numerical system we obtain. This numerical system has at least two symmetries:

$$\frac{m}{n} + \frac{k}{\ell}\sqrt{2} \longrightarrow \frac{m}{n} + \frac{k}{\ell}\sqrt{2} \quad \text{and} \quad \frac{m}{n} + \frac{k}{\ell}\sqrt{2} \longrightarrow \frac{m}{n} - \frac{k}{\ell}\sqrt{2}$$

If the solutions of any polynomial equations, such as $x^3 - x + 1 = 0$, or $x^3 = 2$, are not rational numbers, then we can adjoint them to the rational numbers.

The resulting numerical systems (called number fields) have symmetries which form a group (called the Galois group of the number field)

What Galois has done was bring the idea of symmetry, intuitively familiar in geometry, to the forefront of number theory

Formulas for solutions of equations of degree 3 and 4 were discovered in the early 16th century. Prior to Galois, mathematicians had been trying to find a formula for the solutions of an equation of degree 5 for almost 300 years, to no avail

The question of describing the Galois group turns out to be much more tractable than the question of writing an explicit formula for the solutions

Galois was able to show that a formula for solutions in terms of radicals (square roots, cube roots, and so on) exists if and only if the corresponding Galois group had a particular attribute, which is not present for degree 5 or higher.

Galois' work is a great example of the power of mathematical insight. He hacked the problem!

Galois did not solve the problem of finding a formula for solutions of polynomial equations. He reformulated it, bent and warped it, looked at it in a totally different light.

His brilliant insight has forever changed the way people think about numbers and equations

150 years later, Langlands took these ideas much further. He came up with revolutionary insights tying together the theory of Galois groups and another area of mathematics called harmonic analysis

What followed was the beginning of a groundbeaking theoy that forever changed the way we think about mathematics. The Langlands Program was born

Chapter 8. Magic Numbers

In chapter 2, we saw that representations of a group named $SU(3)$ govern the behavior of elementary particles

The focus of the Langlands program are the representations of the Galois group of symmetries of a number field. Representations of the Galois group form the "source code" of a number field, carrying all essential information about numbers.

This information can be extracted from objects of an entirely different nature: automorphic functions. Automorphic functions are more sophisticated versions of the familiar harmonics: $\sin x, \cos x, \ldots$

In 1986, it was shown that Fermat's Last Theorem follows from something called the Shimura-Taniyama-Weil conjecture, which was not proved at that time

The Shimura-Taniyama-Weil conjecture may be viewed as a special case of the Langlands Program, thus providing an excellent illustration of the unexpected connections predicted by the Langlands Program

The Shimura-Taniyama-Weil conjecture is concerned with a class of algebraic equations in two variables, e.g. : $y^2 + y = x^3 - x^2$. Here, x and y can be natural numbers, or integers, rational numbers, real numbers, complex numbers, ...

How about keeping things finite and look for solutions in the set $\{0, 1, 2, \ldots N - 1\}$. It is important that we restrict N to be a prime number, which we shall rename as p. (In this case, the set is a number field)

Be careful though. We have to do the multiplication and addition "modulo" p

Key Question

How does the number of solutions of this equation, taken modulo p, depend on p?

The number S_p of solutions modulo p is between 1 and p^2. Let us define $a_p = p - S_p$ which could be a positive or negative integer.
(If you know a_p, you know the number of solutions exactly: $S_p = p - a_p$)

The Fibonacci numbers (Warming Up)

1, 1, 2 , 3 , 5 , 8 , 13 , 21 , 34 , ...

$F_1 = 1, F_2 = 1, F_3 = F_1 + F_2, F_4 = F_3 + F_2, \quad \cdots, \quad F_n = F_{n-1} + F_{n-2}$

Is there an explicit formula for F_n?

$q + q(q + q^2) + q(q + q^2)^2 + q(q + q^2)^3 + q(q + q^2)^4 + \ldots$

$q + q^2 + 2q^3 + 3q^4 + 5q^5 + 8q^6 + 13q^7 + \ldots$

$F_n =$ coefficient of q^n

Brace Yourself

$q(1-q)^2(1-q^{11})^2(1-q^2)^2(1-q^{22})^2(1-q^3)^2(1-q^{33})^2(1-q^4)^2(1-q^{44})^2\cdots$

$(1-q)^2 = 1-2q+q^2, \qquad (1-q^{11})^2 = 1-2q^{11}+q^{22}, \qquad \ldots$

$q-2q^2-q^3+2q^4+q^5+2q^6-2q^7-2q^9-2q^{10}+q^{11}-2q^{12}+4q^{13}+\cdots$

Eichler 1954

$a_2=-2, a_3=-1, a_5=1, a_7=-2, a_{11}=1, a_{13}=4, \ldots, a_p=$ coefficient of $q^p \ldots$

$S_2 = 2+a_2 = 2+2 = 4;\ S_3 = 3+a_3 = 3-1 = 2;\ S_5 = 5-a_5 = 5-1 = 4, \ldots$

We started out with what looked like a problem of infinite couplexity: counting solutions of the cubic equation $y^2+y=x^3-x^2$ modulo p.

And yet, all information about this problem is contained in a single line $q-2q^2-q^3+2q^4+q^5+2q^6-2q^7-2q^9-2q^{10}+q^{11}-2q^{12}+4q^{13}+\cdots$

The magnificent insight of Eichler was that the seemingly random numbers of solutions of a cubic equation modulo primes come from a single generating function, which obeys an exquisite symmetry (see the picture on p. 90)

The Shimura-Taniyama-Weil conjecture is a generalization of Eichler's result. It says that for any cubic equation like the one above, the numbers of solutions modulo primes are the coefficients of a "modular form"

The link between the Shimura-Taniyama-Weil conjecture and Fermat's Last Theorem

Starting from a solution of the Fermat equation, one constructs a certain cubic equation (stated explicitly in Endnote 16 of this chapter). The number of solutions of this cubic equation modulo primes cannot be the coefficients of a modular form whose existence is stipulated by the Shimura-Taniyama-Weil conjecture. Once this conjecture is proved, we conclude that such a cubic equation cannot exist. Therefore, there are no solutions to the Fermat equation.

The Shimura-Taniyama-Weil conjecture may be recast as a special case of the Langlands Program. The cubic equations are replaced by two dimensional representations of the Galois group