Freshman Seminar, Winter 2005
Minutes of the Meetings

Prime Obsession
(John Derbyshire)

March 15, 2005

Contents

1 January 10, 2005

1.1 Prologue

- Is there a general rule or formula for how many primes there are less than a given quantity, *that will spare us the trouble of counting them*? (The Prime Number Theorem PNT, proved in 1896, does this approximately; the Riemann Hypothesis RH, still unproven, does this exactly.)

- The Riemann Hypothesis is now the great white whale of mathematical research. The entire twentieth century was bracketed by mathematicians' preoccupation with it.

- Unlike the Four-Color Theorem, or Fermat's Last Theorem, the Riemann Hypothesis is not easy to state in terms a nonmathematician can easily grasp. (The four-color problem was stated in 1852 and solved in 1976; Fermat's Last 'Theorem' was stated in 1637 and solved in 1994; the Riemann Hypothesis was stated in 1859 and remains unsolved to this day.)

- The odd-numbered chapters contain mathematical exposition. The even-numbered chapters offer historical and biographical background matter. Chapters 1-10 constitute Part I: The Prime Number Theorem; chapters 11-22 constitute Part II: The Riemann Hypothesis.

1.2 Chapter 1—Card Trick

- Divergence of the harmonic series (Proved in the 14th century). Convergence of geometric series' (Look at a ruler). Series with alternating signs. This is a part of *analysis*. (See chapter 5 below for more on series)

- The traditional division of mathematics into subdisciplines: Arithmetic (whole numbers), Geometry (figures), Algebra (abstract symbols), Analysis (limits). The first and last combine to form *analytic number theory*. There are others (For example, set theory, probability, statistics, combinatorics, game theory, dynamical systems, topology, APPLIED MATHEMATICS; and more).

1.3 Chapter 2—The Soil, the Crop

- Riemann (1826–1866) does not seem to have been a good scholar. He had the type of mind that could hold only those things it found interesting, mathematics mostly. At some point during the year 1847 Riemann must have confessed to his father that he was far more interested in math than in theology and his father, who seems to have been a kind parent, gave his consent to mathematics as a career.

- Riemann was extremely shy, very pious, thought deeply about philosophy, and was a hypocondriac never in good health. Outwardly he was pitiable; inwardly, he burned brighter than the sun.

- The year 1857 was Riemann's "breakout year." His 1851 doctoral dissertation is nowadays regarded as a classic of 19th century mathematics, and his 1857 paper was at once recognized as a major contribution. In 1859 he was promoted to full professor at Göttingen, on which occasion he submitted a paper titled "On the Number of Prime Numbers Less Than a Given Quantity." Mathematics has not been quite the same since. (Riemann's scientific achievements, which include fundamental contributions to arithmetic, geometry, and analysis, took place during the period 1851-1866.)

1.4 Chapter 3—The Prime Number Theorem

- Do the primes eventually thin out; is there a biggest prime? ANSWERS: YES; NO (300BCE). Can we find a rule, a law, to describe the thinning-out? (There are 25 primes between 1 and 100; 17 between 401 and 500; 14 between 901 and 1000; 4 between 999,901 and 1,000,000.)

- The Prime Counting Function; overloading a symbol. The number of primes up to a given quantity x (x need not be a whole number) is denoted by $\pi(x)$, and is therefore a *function.* If x is an integer, it is usually denoted by the symbol N.

- The Prime Number Theorem states roughly that: $\pi(N)$ behaves very much like $N/\log N$. Empirically, if you compare N with $N/\pi(N)$, each time N is multiplied by 1000, $N/\pi(N)$ goes up by $\log 1000 =$ about 6.9 (see Table 3-2 on page 39 and Table 3-3 on page 46). This is the 'natural logarithm', to base $e = 2.718\cdots$, not to base 10. (PNT was conjectured by Gauss at the end of the 18th century, and proved by two mathematicians (independently and simultaneously) at the end of the 19th century, using tools developed by Riemann in the middle of the 19th century.)

- Two consequences of PNT: (1) the probability that that the N-th natural number is prime is approximately $1/\log N$; (2) The N-th prime number is approximately of size $N \log N$. To illustrate the second statement, the following table implies that $41 = 13 \log 13$ (approximately), $61 = 19 \log 19$ (approximately) and so forth.

numbers	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
primes	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	57	59	61	67

2 January 17, 2005—Holiday

3 January 24, 2005

3.1 Chapter 4—On the Shoulders of Giants

- The greatest mathematician who ever lived was the first person to whom the truth contained in the PNT occurred—Carl Friedrich Gauss (1777-1855). Theorems and proofs that would have made another man's reputation, Gauss left languishing in his personal diaries. (So much to do; so little time!) Two anecdotes about Gauss: $1 + 2 + \cdots + 100 =?$; and the meaning of the word *chiliad*.

- The other first rank mathematical genius born in the 18th century—Leonhard Euler (1707-1783)—solved the "Basel problem" (chapter 5) and discovered the "Golden Key" (chapter 7). (There is also the 'Russian connection': Peter the great established an Academy in St. Petersburg in 1682 and imported Euler from Switzerland to run it—Russia had just come out of a dark period of its development)

3.2 Chapter 5—Riemann's Zeta Function

- The Basel problem opens the door to the zeta function, which is the mathematical object the Riemann Hypothesis is concerned with. Some of the mathematics essential for understanding RH is: powers, roots, and logs. Interesting fact: Any power of $\log x$ eventually increases more slowly than any power of x.

  What is the Basel problem? Consider first some infinite series.

  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots$ (diverges) $\quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots$ (converges,=2) $\quad 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \cdots$ (converges,= 3/2)

  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots (= \log 2) \quad 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \cdots (= 2/3) \quad 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} \cdots (= 3/4)$

  The Basel problem (posed in 1689) is: What is the *exact* value of $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \cdots$?

  The answer (Euler 1735): $\pi^2/6$. He also showed that

  $1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} \cdots = \pi^4/90 \quad 1 + \frac{1}{64} + \frac{1}{3^6} + \frac{1}{4^6} \cdots = \pi^6/945$, and so forth.

In summary, Euler found the exact value of $1+\frac{1}{2^N}+\frac{1}{3^N}+\frac{1}{4^N}\cdots$ for every even $N=2,4,6,\ldots$. However, to this day, no one knows the exact value of this series for any odd value of N, $N=3,5,7,\ldots$.

- Replace the exponent $N=2$ in the Basel problem by any (for the moment *real*) number s to get the zeta function $\zeta(s)$. The series defining the zeta function converges as long as $s>1$ (s need not necessarily be a whole number) but it diverges for $s=1$. It appears that the zeta function also diverges for any $s<1$ (since the terms are bigger than the corresponding terms for $s=1$) and it behaves like $1/(s-1)$ for $s>1$ (see the graph on page 80). Thus, the domain of the zeta function is the set of all (real) numbers greater than 1. Right? Wrong! (Chapter 9)

4 January 31, 2005

Two announcements:

1. Our author, John Derbyshire appeared on C-span yesterday as one of four panelists discussing Tom Wolfe's latest novel: I am Charlotte Simmons. (It is about the wild side of college life!).

2. Please visit my web page www.math.uci.edu/ brusso

4.1 Chapter 6—The Great Fusion

- The Riemann Hypothesis was born out of an encounter between "counting logic," and "measuring logic." Examples: Did the new millenium start on January 1, 2000 (counting logic) or January 1, 2001 (measuring logic)?; Do you turn 21 on your 21st birthday (measuring logic) or on the first day of your 21st year (counting logic)? It arose when some ideas from *arithmetic* (counting) were combined with some from *analysis* (measuring) to form a new branch of mathematics, *analytic number theory.* The great fusion between arithmetic and analysis came about as the result of an inquiry into prime numbers.

- Analysis dates from the invention of calculus by Newton and Leibnitz in the 1670s. Arithmetic, by contrast with analysis, is widely taken to be the easiest, most accessible branch of math. Be careful though—it is rather easy to state problems that are ferociously difficult to prove (e.g., Goldbach conjecture, Fermat's Last 'Theorem'). The Goldbach conjecture asks if every even integer bigger than 2 can be written as the sum of two (odd) primes. This is yet another still unsolved problem with absolutely no real-life application. A problem that can be stated in a few plain words, yet which defies proof by the best mathematical talents for decades (centuries!), has an irresistible attraction for most mathematicians. Even failed attempts can generate powerful new results and techniques. And there is, of course, the Mallory factor.

- Euler proved the Golden Key in 1737. One hundred years later, it came to the attention of Dirichlet, who combined it with Gauss's work on congruences (Clock arithmetic!) to answer an important question about prime numbers, generally considered to be the beginning of analytic number theory (How many primes are there in an arithmetic progression?).

- Gauss and Dirichlet were Riemann's two mathematical idols. If it was Riemann who turned the key, it was Dirichlet who first showed it to him and demonstrated that it *was* a key to something or other; and it is to Dirichlet that the immortal glory of inventing analytic number theory properly belongs.

4.2 Chapter 7—The Golden Key, and an Improved Prime Number Theorem

- Both primes and the zeta function were of interest to Riemann. By yoking the two concepts together, by turning the Golden Key, Riemann opened up the whole field of analytic number theory.

- The Golden Key is just a way that Euler found to express the sieve of Eratosthenes (230 BCE) in the language of analysis. ("You can't beat going to the original sources.") The significance of the Golden Key will be not seen until it is "turned". To prepare for this turning, a little bit of calculus is needed, namely, the basics of differentiation and integration! The Golden Key expresses the Zeta-function, a sum involving all positive integers, in terms of a product involving only prime numbers.

NOTE: The next four items were not discussed today. However, they were discussed on February 7.

- Derivatives give you the gradient (rate of change) of a function—integrals give you the area under graphs.
- An important function, called the "log integral function," and denoted by $Li(x)$, is defined as the area from 0 to x under the graph of $1/\log t$. (Remark for specialists: The derivative of $Li(x)$ is $1/\log x$, which recall is the probability that a whole number in the neighborhood of x is a prime number.)
- The PNT states that $\pi(N)$ behaves very much like $N/\log N$. It is also true that $Li(N)$ behaves very much like $N/\log N$. The improved PNT states that $\pi(N)$ behaves much more like $Li(N)$ than it does like $N/\log N$. The exact formula for $\pi(x)$ (stated by Riemann in 1859) leads off with $Li(x)$.
- Up to at least $N = 100$ trillion, $Li(N)$ is larger than $\pi(N)$. Is $Li(x)$ always bigger than $\pi(x)$? Surprisingly, NO!

4.3 Chapter 8—Not Altogether Unworthy

The main point of interest of this chapter is the part about Chebyshev, whose name, like that of Shakespeare, has numerous spellings, so is a data retrieval nightmare. However, of the three results and two remarks that were made below, I discussed only one of them today (Bertrand's postulate). On February 7, I discussed the fact that Chebyshev's methods were "elementary." I did not mention his other results or his "bias."

- Anyone who wanted to do serious mathematics in the 1840s needed to be in Paris or Berlin. Riemann studied for two years in Berlin under Dirichlet, but returned to Göttingen in 1849 to pursue his doctorate under Gauss. After Riemann's final examination, Gauss drooled: "A substantial and valuable work, which does not merely meet the standards required for a doctoral dissertation, but far exceeds them." Riemann's doctoral thesis is a masterpiece.
- Pafnuty Chebyshev in St. Petersburg made two significant contributions between Dirichlet's picking up the Golden Key in 1837 and Riemann's turning it in 1859. In 1849, a conditional PNT (the condition was removed a half century later!); in 1850, "Bertrand's postulate" (between any number and its double, there is guaranteed to be at least one prime number) and a crude estimate for the difference between $\pi(N)$ and $N/\log N$.
- Chebyshev's methods were elementary, Riemann's were not, nor were the orignial proofs of PNT (1896). An elementary proof of PNT was not produced until 1949 (Atle Selberg and Paul Erdös). (Chebyshev has a *bias* named after him.)
- In Riemann's doctoral dissertation, the "Riemann integral" occurs, now taught as a fundamental concept in calculus courses. His habilitation lecture (second doctoral degree) was on the foundations of geometry. The ideas contained in this paper were so advanced that it was decades before they became fully accepted, and 60 years before they found their natural physical application, as the mathematical framework for Einstein's General Theory of Relativity. That great habilitation lecture is as much a philosophical document as a mathematical one.
- From the death of Gauss to the death of Dirichlet was four years, two months, and twelve days. In that span, Riemann lost not only the two colleagues he had esteemed above all other mathematicians, but also his father, his brother, and two of his sisters. During this time, Riemann's star in the world of mathematics had been rising. It was therefore not very surprising that the authorities selected Riemann as the second successor of Gauss's professorship in 1859. Two weeks later, he was appointed a corresponding member of the Berlin Academy, leading to his famous paper containing his Hypothesis.

5 February 7, 2005

5.1 Chapter 9—Domain Stretching

- The Riemann Hypothesis states: **All non-trivial zeros of the zeta function have real part one-half**. What is a zero of a function? What are the zeros of the zeta function? When are they non-trivial. After we answer these questions we'll move on to "real part one-half." A "zero" of a function is a number a such that the function has the value zero at a. In other words, if you graph the function, its zeros are the numbers on the x-axis at which the function touches crosses the x-axis. A good example is the function $\sin x$, which has zeros at $x = 0, \pi, -\pi, 2\pi, -2\pi, \ldots$

- An infinite series might define only part of a function; in mathematical terms, an infinite series may define a function over only part of its domain. The rest of the function might be lurking away somewhere, waiting to be discovered by some trick. The example here is $S(x) = 1 + x + x^2 + x^3 + \cdots$, which converges for $-1 < x < 1$ and equals $1/(1-x)$ for those values of x. Since $1/(1-x)$ makes sense for all numbers except $x = 1$, this shows that the domain of $S(x)$ is larger than $-1 < x < 1$. Another good example: the Gamma function and the factorial symbol. If you define $H(x) = \int_0^\infty e^{-1} t^{x-1}\, dt$, then one has $H(2) = 1$, $H(3) = 2$, $H(4) = 6$, $H(5) = 24, \ldots$, in fact for every positive integer m, $H(m) = (m-1)! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (m-2)(m-1)$.

- In addition to arguments greater than 1, the zeta function has values for all arguments less than 1. This extension of the zeta function is done in two steps: first to all arguments between 0 and 1 (by changing signs in the series), and then to all negative arguments (by using a deep formula in Riemann's famous 1859 paper). The extended zeta function has the value zero at every negative even number. These are the trivial "zeros" of the zeta function.

- "Harmonic series ... prime numbers ... zeta ... This whole field (of analytic number theory) is dominated by the log function."

6 February 14, 2005

6.1 Reading List for Term Paper/Final Exam

To earn your grade in this freshman seminar, I am asking you to do one of the following. The reports are due at the latest March 21, 2005.

1. Read a few chapters of the following two books, which, together with the book by Derbyshire, were the motivation for this seminar. (Since I am following Derbyshire's book for my lectures, you cannot use it for your project.) Then write a minimum of 3 to 5 page report on what you have read or what impression it made on you. The books are

   - The music of the primes, by Marcus du Sautoy
   - The Riemann Hypothesis, by Karl Sabbagh

   I have a copy of each of these books if you would like to borrow it.

2. Read a few chapters of a book of your choice which is related either to the history of mathematics or to a particular mathematical topic. Then write a minimum of 3 to 5 pages report on what you have learned. Some examples which are already being used are "A beautiful mind", by Sylvia Nasar, and "The golden mean".

3. Choose one of the ten books which I have put on reserve in the Science Library. These books can be borrowed for three days and are the following.

   - Four colors suffice, by Robin Wilson
   - Fermat's Enigma, by Simon Singh

- The Hilbert Challenge, by J. Gray
- On Turing, by John Prager
- Prisoner's Dilemma, by W. Poundstone
- Emmy Noether, by A. Dick
- The constants of nature, by John Barrow
- Math through the ages, by William Berlinghoff
- The mathematical century, by Piergiorgio Odifreddi
- Mathematics and War, by John. A. Adam

6.2 Chapter 10—A Proof and a Turning Point

- Riemann had a strongly visual imaginati–*///////////9-*/39+leaped to results so powerful, elegant, and fruitful that he could not always force himself to pause to prove them. The 1859 paper is therefore revered not for its logical purity, and certainly not for its clarity, but for the sheer originality of the methods Riemannn used, and for the great scope and power of his results, which have provided, and will yet provide, Riemann's fellow mathematicians with decades of research.

- If the Riemann Hypothesis were true, the PNT would follow as a consequence. However, RH is much stronger than PNT, and the latter was proved using weaker tools. There were several significant landmarks between Riemann's paper in 1859 and the proof of PNT (in 1896). The main significance of Riemann's paper for the proof of the PNT is that it provided the deep insights into analytic number theory that showed the way to a proof.

- One of the landmarks alluded to above was the following: In 1895 the German mathematician Hans von Mangoldt proved the main result of Riemann'spaper, which states the connection between $\pi(x)$ and the zeta function. It was then plain that if a certain assertion much weaker than the Riemann Hypothesis could be proved, the application of the result to von Mangoldt's formula would prove the PNT. (N. B. Elaboration of the social, historical, and mathematical background to this 'landmark,' as well as the others is found in paragraphs IV-VI, pages 157–165).

- The PNT follows from a much weaker result (than RH), which has no name attached to it: **All non-trivial zeros of the zeta function have real part less than one.** This result was proved in 1896 simultaneously and independently by Jacques Hadamard (a Frenchman) and Charles de la Vallée Poussin (a Belgian). This established the PNT using the Riemann-von Mangoldt formula.

- If the PNT was the great white whale of number theory in the 19th century, RH was to take its place in the 20th, and moreover was to cast its fascination not only on number theorists, but on mathematicians of all kinds, and even on physicists and philosophers. There is also the neat coincidence of the PNT being first thought of at the end of one century (Gauss, 1792), then being proved at the end of the next (Hadamard and de la Vallée Poussin, 1896). The attention of mathematicians turned to tRH, which occupied them for the following century—which came to its end without any proof being arrived at. (N.B. In an ironic parallel with the current occupant of the White House, that led inquisitive generalists to write books about the PNT and RH at the beginning of the next century.)

- By the later 19th century the world of mathematics had passed out of the era when really great strides could be made by a single mind working alone. Mathematics had become a collegial enterprise in which the work of even the most brilliant scholars was built upon, and nourished by, that of living colleagues. One recognition of this fact was the establishment of periodic International Congresses of Mathematicians, with PNT among the highlights of the first meeting in 1897 in Zürich. There was a second Congress in Paris in 1900. The Paris Congress will forever be linked with the name of David Hilbert, a German mathematician working at Göttingen, the university of Gauss, Dirichlet, and Riemann, for his address on the mathematical challenges of the new century, RH being the most prominent among them.

6.3 Chapter 11—Nine Zulu Queens Ruled China

- We know what the trivial zeros of the zeta function are. What are the non-trivial zeros? For this we need to know about complex numbers.

- Mathematicians think of numbers as a set of nested Russian dolls. The inhabitants of each Russian doll are honorary inhabitants of the next one out. In $\mathbf{N}$ you can't subtract; in $\mathbf{Z}$ you can't divide; in $\mathbf{Q}$ you can't take limits; in $\mathbf{R}$ you can't take the square root of a negative number. With the complex numbers $\mathbf{C}$, nothing is impossible. You can even raise a number to a complex power; therefore, in the zeta function, the variable s may now be a complex number, and the Riemann hypothesis now makes sense: it asserts that the non-trivial zeros of the zeta function all lie on the vertical line whose horizontal coordinate is equal to $1/2$.

- While the real numbers can be spread out for inspection on a line, the complex numbers need a plane. Every complex number has an *amplitude* and a *modulus*.

7 February 21, 2005—Holiday

8 February 28, 2005

From this point one, the "bullets" in boldface will designate the ones which were actually discussed in class.

8.1 Chapter 12—Hilbert's Eighth Problem

- **There are anecdotes about Hilbert in the beginning of this chapter. Any book about Hilbert could be added to the list of possibilities for a term paper in this seminar.**

- **Since 1896 it was known, with mathematical certainty, that, yes indeed, $\pi(N)$ could be approximated arbitrary closely by $N/\log N$. Everyone's attention now focused on the nature of the approximation—What is the error term? Riemann did not prove the PNT, but he strongly suggested it was true, and even suggested an expression for the error term. That expression involved all the non-trivial zeros of the zeta function. One of the questions before us is: what exactly is the relation between the zeros of the zeta function and the prime number theorem. We hope to answer this before the end of the quarter!**

- **At the time of Hilbert's address in 1900, the following was known: all non-trivial zeros lie in the *critical strip*. (RH asserts that they all lie on the *critical line*.) The non-trivial zeros are symmetric about the real axis and about the critical line.**

- To all intents and purposes, the mathematicians of the 19th century left it to those of the 20th to take on Bernhard Riemann's tremendous and subtle conjecture. The story of the Riemann Hypothesis in the 20th century is not a single linear narrative, but a number of threads, sometimes crossing, sometimes tangling with each other.

- **For most of its development, mathematics has been firmly rooted in number. In 20th century math the objects that had been invented to encapsulate important facts about number *themselves became the objects of inquiry*. Mathematics broke free from its mooring in number and soared up to a new level of abstraction. (Example: classical analysis, functional analysis.)**

- These momentous developments have not yet been reflected in mathematics education. A bright young American turning up for a first class as a college math major learns math pretty much as it was known to the young Gauss.

- The story of the Riemann Hypothesis in the 20th century is the story of an obsession that gripped most of the great mathematicians of the age sooner or later. This was not an era short of challenging problems: Fermat's Last Theorem, Four Color Theorem, Goldbach's Conjecture. The Riemann Hypothesis soon came to tower over them all.
- **A number of threads developed during the course of the century—different approaches to investigating the Hypothesis, each originated by some one person, then carried forward by others, the threads sometimes crossing and tangling with each other. (Computational, algebraic, physical, analytic)**
- **The "first" 15 zeros were published in 1903, none contradicting RH. Their real parts were equal to 1/2, as Riemann had hypothesized; but the imaginary parts showed no apparent order or pattern.**

8.2 Chapter13—The Argument Ant and the Value Ant

- The study of "functions of a complex variable" is one of the most elegant and beautiful branches of higher mathematics (both in pure and applied mathematics).
- One of the most beautiful identities in all of mathematics is $e^{\pi i} = -1$. "With complex numbers you can do anything (squares, powers, series, logarithms)."
- **Earlier, we extended the zeta function to all real numbers s except $s = 1$. With s allowed to be a complex number, the zeta function is now defined for every complex number, except again, $s = 1$.**
- One of Riemann's ingenious inventions was that of what is now called a "Riemann surface". Among other things, Riemann surfaces provided a visual aid for studying complex functions. Those "other things" included the connections of complex functions to algebra and topology, two key growth areas of 20th century math.
- **"Figure 13-6 (on page 213) is really the heart of this book." It shows paths in the complex plane which the zeta function sends to either the horizontal axis or the vertical axis. Hence the points of intersection of these paths are examples of 'zeros' of the zeta function. The Riemann Hypothesis shines bright and clear in this diagram—the points of intersection (there are 5 of them in the diagram) all lie halfway between the vertical lines over 0 and 1. (Figure 13-8 on page 220 gives the same information from another vantage point)**
- There is a rule for the average spacing of zeros at height T in the critical strip: it is approximately $2\pi/\log(T/2\pi)$.
- **RH states that all the non-trivial zeros of the zeta function lie on the critical line. All the zeros shown in Figure 13-6 do indeed lie on that line. (Of course, that doesn't prove anything.) The zeta function has an infinite number of non-trivial zeros, and no diagram can show all of them. Do we know whether the trillionth one (for example) lies on the critical line?**

8.3 Chapter 14—In the Grips of an Obsession

- **Just half a generation after Hilbert, three men (two British and one German) stood out as pioneers in the early assaults on RH (G. H. Hardy, J. E. Littlewood, and E. Landau).**
- In *analysis*, the most fertile field of 19th century mathematics, the British were nearly invisible. It was Hardy, more than anyone else, who awoke English pure mathematics from its long slumber.

- **Hardy, an eccentric, wrote an essay, *A Mathematician's Apology* in 1940 (you can add this to the list of possible sources for a term paper in this seminar), in which he described his own life as a mathematician. Hardy stories include "six New-Year wishes"; and "I proved the Riemann Hypothesis."**

- Hardy is best known for two great collaborations, with Ramanujan, and with Littlewood. (A book about Ramanujan would also qualify for a term paper in this seminar)

- **Landau is well-known for his remark about Emmy Noether: "I can testify that Emmy is a great mathematician, but that she is female, I cannot swear,"** and for his two volume work (published in 1909) "Handbook of the Theory of the Distribution of the Prime Numbers." It was from this "Handbook" that both Hardy and Littlewood became infected with the RH obsession.

- **In 1914, each of Hardy and Littlewood came out with results of major importance for RH.**
  Hardy: Infinitely many of the non-trivial zeros of the zeta function have real part 1/2.
  Littlewood: $Li(x) - \pi(x)$ changes from positive to negative and back infinitely often.
  (See the end of Chapter 7.) Where is the first 'Littlewood violation'? This question led to the largest number ever to emerge naturally from a mathematical proof up to that time.

- A consequence of RH (due to von Koch 1901): If RH is true, then the error term $\pi(x) - Li(x)$ is of magnitude no more than $\sqrt{x}\log x$, for large x.

9 March 7, 2005

9.1 Chapter 15—Big Oh and Möbius Mu

- Big Oh is a convenient mathematical notation, invented by Landau, for comparing functions (Figure 15-3 on page 243 depicts von Koch's result. I won't make use of this notation).

- Number theory is now densely populated with results that begin "If RH is true, then ...". If it turns out that RH is false, quite large parts of number theory will have to be rewritten. Analogs of von Koch's results, which do not depend on RH being true, are much uglier! However, they do have the advantage that we know they are true, unconditionally.

- Riemann actually stated (without proof) that the error term was $\sqrt{x}$. Given the tools at his disposal, the state of knowledge in the field, and the known numerical facts at that time (1859), this must still count as intuition of breathtaking depth. A consequence of von Koch's result is that, if RH is true, the error term is no greater than x^a (for large x) where a is any number strictly greater than 1/2 (but arbitrarily close to 1/2).

- **The natural numbers (1,2,3,...) can be put into three categories, which we will call '0,+1,-1' as follows: if n is divisible by a square factor, put n in the class we are calling '0' and write $\mu(n) = 0$ for this n; if n is a prime or is divisible by an odd number of primes, put it in the class we are calling '-1' and write $\mu(n) = -1$ for this n; finally, if n is divisible by an even number of primes, put it in the class we are calling '+1', and write $\mu(n) = 1$ for this n. (For convenience, we put 1 in the class called '1'.) Every natural number n belongs to one of these three classes so has a value $\mu(n)$ attached to it which is either 0, +1, or -1. Here is the point of all this nonsense: the reciprocal of the zeta function, that is, $1/\zeta(s)$ is the infinite sum of the terms $\mu(n)/n^s$, that is**

$$\frac{1}{\zeta(s)} = 1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{10^s} - \frac{1}{11^s} - \frac{1}{13^s} + \cdots$$

- Consider the cumulative values of the Möbius function $\mu(n)$, that is, let $M(k) = \mu(1) + \mu(2) + \mu(3) + \cdots + \mu(k)$. Here is another formulation of RH (that is, this statement is equivalent to RH; both are true, or both are false): $M(k)$ is of the order of magnitude no more than $\sqrt{k}$, for large k.

9.2 Chapter 16—Climbing the Critical Line

- In 1930, on the occasion of his retirement at the age of 68, Hilbert delivered the second great public speech of his career. His last six words, "We must know, we shall know", are the most famous that Hilbert ever spoke, and among the most famous in the history of science.

- **By April 1933, the Nazis had almost total control of Germany. One of their first decrees was intended to bring about the dismissal of all Jews (with some exceptions) from the civil service (which included University Professors). Uniformed storm troopers prevented Landau's students from entering the lecture hall for his calculus class. Between April and November of 1933, Göttingen as a mathematical center was gutted. The mathematicians fled, most eventually finding their way to the U.S.**

- Those years of the early 1930s, before the darkness fell, brought forth one of the most romantic episodes in the history of RH, the discovery of the Riemann-Siegel formula. Carl Ludwig Siegel was one of the few mathematicians to persevere after examining Riemann's private mathematical papers. Others had been defeated by the fragmentary and disorganized style of Riemann's jottings, or else they lacked the mathematical skills needed to understand them. To explain Siegel's contribution, it is necessary to return to the computational thread started at the end of chapter 12.

- **Table 16-1 on page 258 shows the incremental progress in verifying RH from 1903 to 1986, from 15 non-trivial zeros to 1.5 billion of them. As of August 1, 2002, 100 billion zeros have been shown to satisfy RH.** Besides the actual number of zeros, also of interest is their height up the critical line, and the accuracy (number of decimal places) in their value. There is a formula for the number $N(T)$ of zeros up to a given height T: namely, it is approximately $(T/2\pi)\log(T/2\pi) - T/2\pi$.

- **Alan Turing was a genius famous for the Turing test (a way of deciding whether a computer or its program is intelligent), the Turing machine (a very general, theoretical type of computer, a thought experiment used to tackle certain problems in mathematical logic), and the Turing prize for achievement in computer science (similar to the Fields Medal in mathematics, and the Nobel prize in other fields). Turing was fascinated by RH, believed it false and spent much of his time trying to find a zero off the critical line. Besides the book on Turing which was suggested above as a possible project topic for this seminar, another is *Alan Turing: The Enigma*, 1983, by Andrew Hodges.**

- Before 1935, all calculations of zeros was done only with paper, pencil, and books of mathematical tables. Bernhard Riemann, in the background work for his famous 1859 paper, had developed an improved method (over the method used before 1935) for working out the zeros—and had actually implemented it and computed the first three zeros for himself. The discovery, by Siegel, of Riemann's formula, fine-tuned and published by Siegel to become the Riemann-Siegel formula, made work on the zeros much easier. All significant research depended on it up to the mid 1980s.

9.3 Chapter 17—A Little Algebra

- Modern math is very *algebraic*. Problems from other fields are often translated into algebra, solved there, and then translated back to the original field. **The algebra relevant to RH concerns *field theory*. A *field* is an algebraic structure in which there are defined (abstract) operations of addition, subtraction, multiplication, and division. The rationals Q, the real numbers R, the complex numbers C are examples of fields with infinitely many elements. The fields of interest to RH have only finitely many elements and are associated with prime numbers.**

- **Field theory opened up a new approach to the Riemann Hypothesis** (Emil Artin, 1921). Andre Weil (1942) conjectured an abstract RH based on finite fields (The classical RH is based on the familiar field of rational numbers **Q**). In 1973 the Belgian mathematician Pierre Deligne proved the Weil conjectures, essentially completing a program initiated by Artin, and earning himself a Fields

Medal. Whether the techniques developed to prove these analogues of RH for these very abstruse fields can be used to solve the classical RH is not known. However, they were used by Andrew Wiles in 1994 to solve Fermat's Last Theorem.

- **Another approach to RH is based on *operator theory*, another algebraic topic. Operators are best understood by their representation as matrices, with the associated mathematical constructs: characteristic polynomial, eigenvalues, trace.** These are properties of an operator, and not of the matrix that represents the operator.

- An important class of matrices are the *hermitian* ones, and the main fact about them is that their eigenvalues must be real numbers. RH states that if you write the zeros of the zeta function in the form $1/2 + iz$ then all the $z's$ are real. This suggests the Hilbert-Polya Conjecture: The non-trivial zeros of the Riemann zeta function correspond to the eigenvalues of some Hermitian operator.

- Because Hermitian operators played a significant role in the development of Quantum Mechanics in the 1920s, it is not surprising that Landau posed the following question to Polya: "Can you think of any *physical* reason why the Riemann Hypothesis might be true?".

9.4 Chapter 18—Number Theory Meets Quantum Mechanics

- The Hilbert-Polya conjecture was far ahead of its time and lay there untroubled for half a century. Nevertheless, **the first half of the 20th century was very eventful in physics (splitting of atom, chain reaction, nuclear explosion in 1945).**

- Questions about energy levels of sub-nuclear particles led to a larger class of problems, problems about *dynamical systems*, collections of particles each of which has, at any point in time, a certain position and a certain velocity. By 1950 it became apparent that some of the most interesting dynamical systems were too complicated to yield to exact mathematical analysis, so investigators fell back on statistics: What, on average, is most likely to happen? Key players in this were the nuclear physicists Eugene Wigner and Freeman Dyson, and a central concept was that of a *ramdom matrix.*

- Very large bell curve type random Hermitian matrices proved to be just the ticket for modeling the behavior of certain quantum-dynamical systems, and their eigenvalues turned out to provide an excellent fit for the energy levels observed in experiments. Therefore, these eigenvalues became the subject of intensive study by physicists through the 1960s.

- **A chance encounter between a number theorist (Hugh Montgomery, a graduate student at the time) and a physicist (Freeman Dyson) occurred at Princeton in 1972. Montgomery was investigating the spacing between non-trivial zeros of the zeta function and was asked (politely) by Dyson what he was working on. A formula of Montgomery's about Riemann's zeta function was recognized by Dyson as being strongly related to eigenvalues of random Hermitian matrices. Following this link, much of the recent thinking about RH has been done by physicists** and applied mathematicians, resulting in fewer "rigorous" results (Richard Feynman: "A great deal more is known than has been proved.")

- As opposed to the case for zeta zeros, the statistical law followed by random matrices had been widely studied. Therefore Andrew Odlyzko (AT and T) took up the study of the proposed statistical law for zeta zeros, using high powered computers (1978). The results (1987) were not quite conclusive. "The data presented so far are fairly consistent with the random matrix predictions."

- **It seems that the non-trivial zeros of the zeta function and the eigenvalues of random Hermitian matrices are related in some way. The non-trivial zeros of the zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics.** *What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles?*

10 March 14, 2005

10.1 Chapter 19—Turning the Golden Key

- We now go to the heart of Riemann's 1859 paper, outlining the main logical steps. The reward will be a result of great beauty and power, from which flows everything—the Hypothesis, its importance, and its relevance to the distribution of prime numbers.

- The prime counting function $\pi(x)$ is an example of a step function. Another example which Riemann invented is the "J" function: $J(x) = \pi(x) + \frac{1}{2}\pi(\sqrt{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \frac{1}{4}\pi(\sqrt[4]{x}) + \ldots$. Inverting this relation (using something called "Möbius inversion"), leads to $\pi(x) = J(x) - \frac{1}{2}J(\sqrt{x}) - \frac{1}{3}J(\sqrt[3]{x}) - \frac{1}{5}J(\sqrt[5]{x}) + \frac{1}{6}J(\sqrt[6]{x}) - \frac{1}{7}J(\sqrt[7]{x}) + \ldots$. We now have $\pi(x)$ expressed in terms of $J(x)$. That is a wonderful thing, because Riemann found a way to express $J(x)$ in terms of the zeta function $\zeta(x)$.

- Starting with his Golden Key (the fancy way to write out the sieve of Erathostenes) and using logarithms and calculus, Riemann arrived at a wonderful result, a relation between the zeta function and the J function, namely $\frac{1}{s}\log\zeta(s) = \int_0^\infty J(x)x^{-s-1}\,dx$. Using one more inversion process on this to express $J(x)$ in terms of the zeta function, together with the fact that $\pi(x)$ is expressed in terms of $J(x)$, tells us that $\pi(x)$ can be expressed in terms of the zeta function. Thus, all the properties of the π function will be found encoded somehow in the properties of the zeta function.

10.2 Chapter 20—The Riemann Operator and Other Approaches

- The facts that the non-trivial zeros of the zeta function resemble the eigenvalues of some random Hermitian matrix, and that the operators represented by such matrices can be used to model certain dynamical systems in quantum physics led to the following questions: Is there an operator whose eigenvalues are precisely the zeta zeros? If there is, what dynamical system does it represent? Could that system be created in a physics lab? And if it could, would that help to prove RH? Sir Michael Berry's answer in 1986 to one of these questions was: a (semi-classical) *chaotic* system.

- That pure number theory—ideas about the natural numbers and their relations with each other—should have relevance to sub atomic physics is not at all that surprising: quantum physics has a much stronger arithmetical component than classical physics, since it depends on the idea that matter and energy are not infinitely divisible.

- The two-body problem is managable and predictable. This is not so for more complicated problems, like the three-body problem. The only way to get solutions is by extensive numerical calculation, leading to approximations. In fact, these solutions are sometimes *chaotic* (Poincaré, 1890).

- Nothing much happened in chaos theory for several decades, mainly because mathematicians had no way to do number-crunching on the scale required to analyze chaotic results. This changed in the 1960s, and chaos theory is now a vast subject embracing many different subdisciplines within physics, mathematics, and computer science. The beauty of chaos theory is that there are patterns embedded in chaotic systems.

- Although chaos theory was thought by most physicists to be a classical matter, in fact, a certain level of chaos can be observed in quantum-scale dynamical systems.

- Alain Connes constructed an operator whose eigenvalues are the zeta zeros. The space on which this operator acts has the prime numbers built in to it, yet is relevant to actual physical systems, actual assemblies of subatomic particles, and was used by mathematical physicists in the 1990s. Opinions as to the value of Connes's work vary widely.

- Direct approaches to RH: besides analytic number theory, there are algebraic approaches through finite fields and through Riemann operators, the latter connected with physical lines of attack. Indirect approaches: a probabilistic interpretation of the Möbius function approach of chapter 15, and the idea that primes were distributed as randomly as they could be (Cramér's model); an approach through non-deductive logic, as if RH were being decided in a court of law.

10.3 Chapter 21—The Error Term

- In chapter 19 we were told that a sufficiently close study of the zeta function ζ will tell us all we want to know about the prime counting function π. How does this actually work? In particular, how does that "middle man" function $J(x)$ look like when written in terms of ζ?

- The main result of Riemann's celebrated 1859 paper is the formula

$$J(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}.$$

  The first, third, and fourth terms on the right-hand side are easily understood. The second term is the heart of the matter. It is crucial to understanding the distribution of primes.

- The summation indicated in the second term is over all the non-trivial zeros of the zeta function, and this term arises from the inversion process that gives $J(x)$ in terms of $\zeta(x)$. This inversion process is beyond the scope of our seminar (Leap of Faith I), so suffice it to say it is a generalization of the well-know fact that any polynomial function can be expressed in terms of its roots (root=zero!).

- For a fixed real number x, the function $f(z) = x^z$ maps the critical strip in the argument plane onto a circle with center at the origin and radius $\sqrt{x}$ in the value plane. Assuming RH to be true, the non-trivial zeros of the zeta function, denoted generically by ρ, are mapped into points x^{ρ} scattered on this circle. We then use these points as arguments for our old friend the log integral function. (Leap of Faith II: the log integral function $Li(y)$ can be defined, not just for real numbers y, but for complex numbers z.)

- Paragraphs VI and VII of this chapter show the details in the calculation of $J(20)$ using Riemann's formula $J(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}$. Paragraph VIII then uses this formula to calculate $J(\sqrt[N]{1,000,000})$ for $N = 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19$, and thus calculate and analyze $\pi(1,000,000)$ using the formula

$$\pi(1,000,000) = J(1,000,000) - \frac{1}{2}J(\sqrt{1,000,000}) - \frac{1}{3}J(\sqrt[3]{1,000,000}) + \ldots - \frac{1}{19}J(\sqrt[19]{1,000,000})$$

  from chapter 19. Paragraph IX further analyzes this technique for very large values of x.

- All of these calculations have assumed that RH is true. If it is not true, the logic of the chapter falls apart. In the theory of the error term, RH is central.

- The intimate connection has now been revealed between the distribution of prime numbers, as embodied in $\pi(x)$, and the non-trivial zeros of the zeta function, a component of the difference between $\pi(x)$ and $Li(x)$, that is, of the error term in PNT. All of this was revealed by Riemann's dazzling 1859 paper. What are the prospects now, in the fifteenth decade of our efforts to crack RH?

10.4 Chapter 22—Either It's True, or Else It Isn't

- There is a satisfying symmetry about the fact that RH, after 120 years among the mathematicians, has got the attention of the physicists. The distinction between mathematician and physicist was not much made in Riemann's time. And Riemann's own imagination was very much that of a physical scientist.

- The American Institute of Mathematics (AIM) has been a considerable force in assaults on RH during recent years, funding three conferences devoted to RH (in 1996,1998,2002). One of its founders (in 1994) is John Fry (Fry's Electronics). Another privately-funded enterprise similar to AIM is the Clay Mathematical Institute (CMI), started in 1998, whose initial focus was on Hilbert's 1900 address. In 2000, a $7 million fund was established, with $1 million dollars to be awarded for the solution of each of seven great mathematical problems (the Millenium Problems). Of course, RH is among them.

- Hilbert said 75 years ago that RH would be resolved in his lifetime. He was drastically wrong about this. The author of this book believes that a proof of RH, despite occasional advances, is a long way beyond our present grasp. There is a current lull among researchers. The last great spurts were Deligne's proof of the Weil Conjectures in 1973 and the Montgomery-Odlyzko developments of 1972-1987. Mathematicians at the 2002 conference has little to show in the six years since the 1996 conference.

- How do mathematicians feel about the truth or falsity of RH? What do they predict the final answer will be? Among the majority of mathematicians who believe it true, it is the sheer weight of evidence that tells. There are hundreds of theorems that begin, "Assuming the truth of the Riemann Hypothesis,...". They would all come crashing down if RH were to be false. Other mathematicians believe, as Alan Turing did, that RH is probably false.

- What use is RH, if it is true? Would our health, our convenience, our safety be improved? Most mathematicians are motivated not by any thought of advancing the health or convenience of the human race, but by the sheer joy of discovery and the challenge of tackling difficult problems. Despite his outstanding contributions and reputation, Hardy ends his essay with "Judged by all practical standards, the value of my mathematical life is nil."

- Beginning in the late 1970s, prime numbers began to attain great importance in the design of encryption methods for both military and civilian use. Theoretical results, including some of Hardy's, were essential in these developments, which, among other things, alow you to use your credit card to order goods over the internet. A resolution of RH would undoubtedly have further consequences in this field, and act as a spur to further discoveries.