Triple derivations on von Neumann algebras

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Outline

- Introduction
- Varieties of modules
- Derivations
- Ternary Weak Amenability
- Normal and Approximate Ternary Weak Amenability

Introduction

Building on earlier work of Bunce and Paschke, Haagerup showed in 1983 that every derivation of a von Neumann algebra into its predual is inner, and as a consquence that every C^*-algebra is weakly amenable.

Last year, Ho, Peralta, and R initiated the study of ternary weak amenability in operator algebras and triples, defining triple derivations and inner triple derivations into a Jordan triple module.

Inner triple derivations on a von Neumann algebra M into its predual M_* are closely related to spans of commutators of normal functionals with elements of M. (Not really very surprising)

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Two consequences of that work are that finite dimensional von Neumann algebras and abelian von Neumann algebras have the property that every triple derivation into the predual is an inner triple derivation, analogous to the Haagerup result.

This rarely happens in a general von Neumann algebra, but it comes close. We prove a (triple) cohomological characterization of finite von Neumann algebras and a zero-one law for factors.

Main Result (in Words)

For any von Neumann algebra, the linear space of bounded triple derivations into the predual modulo the norm closure of the inner triple derivations has dimension 0 or 1. It is zero if and only if the von Neumann algebra is finite; and it is 1 if the von Neumann algebra is a properly infinite factor.

This is the beginning of a joint project with Robert Pluta. These ideas will be explored further in various contexts (C^*-algebras, Jordan C^*-algebras, ternary rings of operators, noncommutative L^p spaces).

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Modules

Let A be an associative algebra. Let us recall that an A-**bimodule** is a vector space X, equipped with two bilinear products $(a,x) \mapsto ax$ and $(a,x) \mapsto xa$ from $A \times X$ to X satisfying the following axioms:

$$a(bx) = (ab)x, \quad a(xb) = (ax)b, \text{ and, } (xa)b = x(ab),$$

for every $a, b \in A$ and $x \in X$.

The space $A \oplus X$ is an associative algebra with respect to the product

$$(a,x)(b,y) := (ab, ay + bx).$$

Let A be a Jordan algebra. A **Jordan** A-**module** is a vector space X, equipped with two bilinear products $(a,x) \mapsto a \circ x$ and $(x,a) \mapsto x \circ a$ from $A \times X$ to X, satisfying:

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,}$$
$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2,$$

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Jordan triple system

A complex (resp., real) **Jordan triple** is a complex (resp., real) vector space E equipped with a triple product

$$E \times E \times E \to E \qquad (xyz) \mapsto \{x, y, z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one and satisfying the so-called *"Jordan Identity"*:

$$L(a, b)L(x, y) - L(x, y)L(a, b) = L(L(a, b)x, y) - L(x, L(b, a)y),$$

for all a, b, x, y in E, where $L(x, y)z := \{x, y, z\}$.

The Jordan identity is equivalent to

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\},$$

which asserts that the map $iL(a, a)$ is a *triple derivation* (to be defined shortly).

It also shows that the span of the "multiplication" operators $L(x, y)$ is a Lie algebra.

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Jordan triple module

Let E be a complex (resp. real) Jordan triple. A **Jordan triple E-module** is a vector space X equipped with three mappings

$$\{.,.,.\}_1 : X \times E \times E \to X, \quad \{.,.,.\}_2 : E \times X \times E \to X$$
$$\text{and } \{.,.,.\}_3 : E \times E \times X \to X$$

in such a way that the space $E \oplus X$ becomes a real Jordan triple with respect to the triple product $\{a_1 + x_1, a_2 + x_2, a_3 + x_3\} =$

$$\{a_1, a_2, a_3\}_E + \{x_1, a_2, a_3\}_1 + \{a_1, x_2, a_3\}_2 + \{a_1, a_2, x_3\}_3.$$

(PS: we don't really need the subscripts on the triple products)

The Jordan identity

$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\},$
holds whenever exactly one of the elements belongs to X.

In the complex case we have the unfortunate technical requirement that
$\{x, a, b\}_1$ ($=\{b, a, x\}_3$) is linear in a and x and conjugate linear in b
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Every (associative) Banach A-bimodule (resp., Jordan Banach A-module) X over an associative Banach algebra A (resp., Jordan Banach algebra A) is a real Banach triple A-module (resp., A-module) with respect to the "*elementary*" product

$$\{a, b, c\} := \frac{1}{2}(abc + cba)$$

(resp., $\{a, b, c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$), where one element of a, b, c is in X and the other two are in A.

The dual space, E^*, of a complex (resp., real) Jordan Banach triple E is a complex (resp., real) triple E-module with respect to the products:

$$\{a, b, \varphi\}(x) = \{\varphi, b, a\}(x) := \varphi\{b, a, x\} \tag{1}$$

and

$$\{a, \varphi, b\}(x) := \overline{\varphi\{a, x, b\}}, \tag{2}$$

$\forall x \in X, a, b \in E, \varphi \in E^*$.

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Derivations

Let X be a Banach A-bimodule over an (associative) Banach algebra A. A linear mapping $D : A \to X$ is said to be a **derivation** if $D(ab) = D(a)b + aD(b)$, for every a, b in A. For emphasis we call this a **binary (or associative) derivation**.

We denote the set of all continuous binary derivations from A to X by $\mathcal{D}_b(A, X)$.

When X is a Jordan Banach module over a Jordan Banach algebra A, a linear mapping $D : A \to X$ is said to be a **derivation** if $D(a \circ b) = D(a) \circ b + a \circ D(b)$, for every a, b in A. For emphasis we call this a **Jordan derivation**.

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In the setting of Jordan Banach triples, a **triple** or **ternary derivation** from a (real or complex) Jordan Banach triple, E, into a Banach triple E-module, X, is a conjugate linear mapping $\delta : E \to X$ satisfying

$$\delta \{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\} , \qquad (3)$$

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We denote the set of all continuous ternary derivations from E to X by $\mathcal{D}_t(E, X)$.

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In the setting of Jordan Banach triples, a **triple** or **ternary derivation** from a (real or complex) Jordan Banach triple, E, into a Banach triple E-module, X, is a conjugate linear mapping $\delta : E \to X$ satisfying

$$\delta \{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}, \quad (3)$$

for every a, b, c in E.

We denote the set of all continuous ternary derivations from E to X by $\mathcal{D}_t(E, X)$.

Inner derivations

Let X be a Banach A-bimodule over an associative Banach algebra A. Given x_0 in X, the mapping $D_{x_0} : A \to X$, $D_{x_0}(a) = x_0 a - a x_0$ is a bounded (associative or binary) derivation. Derivations of this form are called **inner**.

The set of all inner derivations from A to X will be denoted by $\mathcal{I}nn_b(A, X)$.

When x_0 is an element in a Jordan Banach A-module, X, over a Jordan Banach algebra, A, for each $b \in A$, the mapping $\delta_{x_0,b} : A \to X$,

$$\delta_{x_0,b}(a) := (x_0 \circ a) \circ b - (b \circ a) \circ x_0,\ (a \in A),$$

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Let E be a complex (resp., real) Jordan triple and let X be a triple E-module. For each $b \in E$ and each $x_0 \in X$, we conclude, via the main identity for Jordan triple modules, that the mapping $\delta = \delta(b, x_0) : E \to X$, defined by

$$\delta(a) = \delta(b, x_0)(a) := \{b, x_0, a\} - \{x_0, b, a\} \quad (a \in E), \tag{4}$$

is a ternary derivation from E into X. Finite sums of derivations of the form $\delta(b, x_0)$ are called **inner triple derivations**.

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Ternary Weak Amenability (Ho-Peralta-R)

Proposition

Let A be a unital Banach *-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ and the Jordan product $a \circ b = (ab + ba)/2$. Then

$$\mathcal{D}_t(A, A^*) \subset \mathcal{D}_J^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*).$$

Proposition

Every commutative (real or complex) C*-algebra A is **ternary weakly amenable**, that is $\mathcal{D}_t(A, A^*) = \mathcal{I}nn_t(A, A^*)$ ($\neq 0$ btw).

Proposition

The C*-algebra $A = M_n(\mathbb{C})$ is ternary weakly amenable (Hochschild 1945) and **Jordan weakly amenable** (Jacobson 1951).

Question

Is $C_0(X, M_n(\mathbb{C}))$ ternary weakly amenable?

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The C*-algebra $A = K(H)$ of all compact operators on an infinite dimensional Hilbert space H is **not** ternary weakly amenable.

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The C*-algebra $A = B(H)$ of all bounded operators on an infinite dimensional Hilbert space H is **not** ternary weakly amenable.

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Non algebra results

Theorem

Let H and K be two complex Hilbert spaces with $\dim(H) = \infty > \dim(K)$. Then the rectangular complex Cartan factor of type I, $L(H, K)$, and all its real forms are **not** ternary weakly amenable.

Theorem

Every commutative (real or complex) JB*-triple E is **approximately ternary weakly amenable**, that is, $\mathcal{I}nn_t(E, E^*)$ is a norm-dense subset of $\mathcal{D}_t(E, E^*)$.

Commutative Jordan Gelfand Theory (Kaup,Friedman-R)

Given a commutative (complex) JB*-triple E, there exists a principal $\mathbb{T}$-bundle $\Lambda = \Lambda(E)$, i.e. a locally compact Hausdorff space Λ together with a continuous mapping $\mathbb{T} \times \Lambda \to \Lambda$, $(t, \lambda) \mapsto t\lambda$ such that $s(t\lambda) = (st)\lambda$, $1\lambda = \lambda$ and $t\lambda = \lambda \Rightarrow t = 1$, satisfying that E is JB*-triple isomorphic to

$$\mathcal{C}_0^{\mathbb{T}}(\Lambda) := \{f \in \mathcal{C}_0(\Lambda) : f(t\lambda) = tf(\lambda), \forall t \in \mathbb{T}, \lambda \in \Lambda\}.$$

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Normal ternary weak amenability (Pluta-R)

Corollary

Let M be a von Neumann algebra and consider the submodule $M_* \subset M^*$. Then

$$\mathcal{D}_t(M, M_*) = \mathcal{I}nn_b^*(M, M_*) \circ * + \mathcal{I}nn_t(M, M_*).$$

Note

L^∞ is ternary weakly amenable and **normally ternary weakly amenable**, that is, $\mathcal{D}_t(L^\infty, L^1) = \mathcal{I}nn_t(L^\infty, L^1)$.

Question

Is $L^\infty \otimes M_n(\mathbb{C})$ normally ternary weakly amenable?

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Main Results

Theorem

If M is a properly infinite factor, then the real vector space of triple derivations of M into M_*, modulo the norm closure of the inner triple derivations, has dimension 1.

$$\mathcal{D}_t(M, M_*)/\overline{\mathcal{I}nn_t(M, M_*)} \sim \mathbb{R}$$

Theorem

If M is a von Neumann algebra, then M is finite if and only if every triple derivation of M into M_* is approximated in norm by inner triple derivations.

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