GRADUS AD PARNASSUM

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1 ALGEBRAS

1. Prove Proposition 2: FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE
$$\delta_A(X) = AX - XA.$$
THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION

2. Prove Proposition 3: FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE
$$\delta_A(X) = [A, X] = AX - XA.$$
THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION

3. Prove Proposition 4: FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE
$$\delta_A(X) = AX - XA.$$
THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION

4. Let A, B are two fixed matrices in $M_n(\mathbf{C})$.
Show that the linear mapping
$$\delta_{A,B}(X) = A \circ (B \circ X) - B \circ (A \circ X)$$
is a derivation of $M_n(\mathbf{C})$ with respect to circle multiplication.
(cf. Remark following Theorem 4)

5. Show that $M_n(\mathbf{C})$ is a Lie algebra with respect to bracket multiplication.

6. Show that $M_n(\mathbf{C})$ is a Jordan algebra with respect to circle multiplication.

7. Let us write $\delta_{a,b}$ for the linear mapping $\delta_{a,b}(x) = a(bx) - b(ax)$ in a Jordan algebra. Show that $\delta_{a,b}$ is a derivation of the Jordan algebra by following the outline below. (cf. problem 4 above.)

(a) In the Jordan algebra axiom
$$u(u^2v) = u^2(uv),$$
replace u by $u + w$ to obtain the equation
$$2u((uw)v) + w(u^2v) = 2(uw)(uv) + u^2(wv) \tag{1}$$

(b) In (1), interchange v and w and subtract the resulting equation from (1) to obtain the equation
$$2u(\delta_{v,w}(u)) = \delta_{v,w}(u^2). \tag{2}$$

(c) In (2), replace u by $x + y$ to obtain the equation
$$\delta_{v,w}(xy) = y\delta_{v,w}(x) + x\delta_{v,w}(y),$$
which is the desired result.

2 TRIPLES

8. Prove Proposition 5: FOR TWO MATRICES $A \in M_m(\mathbf{C}), B \in M_n(\mathbf{C})$, WITH $A^* = -A, B^* = -B$, DEFINE $\delta_{A,B}(X) = AX + XB$. THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION. (Use the notation $\langle abc\rangle$ for ab^*c)

9. Prove Proposition 6: FIX TWO MATRICES A, B IN $M_n(\mathbf{C})$ AND DEFINE $\delta_{A,B}(X) = [[A,B],X]$. THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION. (Use the notation $[abc]$ for $[[a,b],c]$)

10. Prove Proposition 7: FIX TWO MATRICES A, B IN $M_{m,n}(\mathbf{C})$ AND DEFINE $\delta_{A,B}(X) = \{A,B,X\} - \{B,A,X\}$. THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION. (Use the notation $\{abc\}$ for $ab^*c + cb^*a$)

11. Show that $M_n(\mathbf{C})$ is a Lie triple system with respect to triple bracket multiplication. In other words, show that the three axioms for Lie triple systems in Table 4 are satisfied if abc denotes $[[a,b],c] = (ab-ba)c - c(ab-ba)$ (a, b and c denote matrices). (Use the notation $[abc]$ for $[[a,b],c]$)

12. Show that $M_{m,n}(\mathbf{C})$ is a Jordan triple system with respect to triple circle multiplication. In other words, show that the two axioms for Jordan triple systems in Table 4 are satisfied if abc denotes $ab^*c + cb^*a$ (a, b and c denote matrices). (Use the notation $\{abc\}$ for $ab^*c + cb^*a$)

13. Let us write $\delta_{a,b}$ for the linear process

$$\delta_{a,b}(x) = abx$$

in a Lie triple system. Show that $\delta_{a,b}$ is a derivation of the Lie triple system by using the axioms for Lie triple systems in Table 4. (Use the notation $[abc]$ for the triple product in any Lie triple system, so that, for example, $\delta_{a,b}(x)$ is denoted by $[abx]$)

14. Let us write $\delta_{a,b}$ for the linear process

$$\delta_{a,b}(x) = abx - bax$$

in a Jordan triple system. Show that $\delta_{a,b}$ is a derivation of the Jordan triple system by using the axioms for Jordan triple systems in Table 4. (Use the notation $\{abc\}$ for the triple product in any Jordan triple system, so that, for example, $\delta_{a,b}(x) = \{abx\} - \{bax\}$)

15. On the Jordan algebra $M_n(\mathbf{C})$ with the circle product $a \circ b = ab + ba$, define a triple product

$$\{abc\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b.$$

Show that $M_n(\mathbf{C})$ is a Jordan triple system with this triple product.

Hint: show that $\{abc\} = 2abc + 2cba$

16. On the vector space $M_n(\mathbf{C})$, define a triple product $\langle abc\rangle = abc$ (matrix multiplication without the adjoint in the middle). Formulate the definition of a derivation of the resulting triple system, and state and prove a result corresponding to Proposition 5. Is this triple system associative?

17. In an associative algebra, define a triple product $\langle abc\rangle$ to be abc. Show that the algebra becomes an associative triple system with this triple product.

18. In an associative triple system with triple product denoted $\langle abc\rangle$, define a binary product ab to be $\langle aub\rangle$, where u is a fixed element. Show that the triple system becomes an associative algebra with this product. Suppose further that $\langle auu\rangle = a$ for all a. Show that we get a unital involutive algebra with involution $a^\sharp = \langle uau\rangle$.

19. In a Lie algebra with product denoted by $[a, b]$, define a triple product $[abc]$ to be $[[a, b], c]$. Show that the Lie algebra becomes a Lie triple system with this triple product.
(Meyberg Lectures, chapter 6, example 1, page 43)

20. Let A be an algebra (associative, Lie, or Jordan; it doesn't matter). Show that the set $\mathcal{D} := \mathrm{Der}\,(A)$ of all derivations of A is a Lie subalgebra of $\mathrm{End}\,(A)$. That is, $\mathcal{D}$ is a linear subspace of the vector space of linear transformations on A, and if $D_1, D_2 \in \mathcal{D}$, then $D_1D_2 - D_2D_1 \in \mathcal{D}$.

21. Let A be a triple system (associative, Lie, or Jordan; it doesn't matter). Show that the set $\mathcal{D} := \mathrm{Der}\,(A)$ of derivations of A is a Lie subalgebra of $\mathrm{End}\,(A)$. That is, $\mathcal{D}$ is a linear subspace of the vector space of linear transformations on A, and if $D_1, D_2 \in \mathcal{D}$, then $D_1D_2 - D_2D_1 \in \mathcal{D}$.

3 SUPPLEMENT (ALGEBRAS AND TRIPLES)

22. In an arbitrary Jordan triple system, with triple product denoted by $\{abc\}$, define a triple product by

$$[abc] = \{abc\} - \{bac\}.$$

Show that the Jordan triple system becomes a Lie triple system with this new triple product.
(Meyberg Lectures, chapter 11, Theorem 1, page 108)

23. In an arbitrary associative triple system, with triple product denoted by $\langle abc \rangle$, define a triple product by

$$[xyz] = \langle xyz \rangle - \langle yxz \rangle - \langle zxy \rangle + \langle zyx \rangle.$$

Show that the associative triple system becomes a Lie triple system with this new triple product.
(Meyberg Lectures, chapter 6, example 3, page 43)

24. In an arbitrary Jordan algebra, with product denoted by xy, define a triple product by $[xyz] = x(yz) - y(xz)$. Show that the Jordan algebra becomes a Lie triple system with this new triple product.
(Meyberg Lectures, chapter 6, example 4, page 43)

25. In an arbitrary Jordan triple system, with triple product denoted by $\{abc\}$, fix an element y and define a binary product by

$$ab = \{ayb\}.$$

Show that the Jordan triple system becomes a Jordan algebra with this (binary) product.
(Meyberg Lectures, chapter 10, Theorem 1, page 94—using different language; see also, Harald Upmeier, Symmetric Banach Manifolds and Jordan C*-algebras, 1985, Proposition 19.7, page 317))

26. In an arbitrary Jordan algebra with multiplication denoted by ab, define a triple product

$$\{abc\} = (ab)c + (cb)a - (ac)b.$$

Show that the Jordan algebra becomes a Jordan triple system with this triple product. (cf. Problem 15)
(Meyberg Lectures, chapter 10, page 93—using different language; see also, Harald Upmeier, Symmetric Banach Manifolds and Jordan C*-algebras, 1985, Corollary 19.10, page 320)

27. Show that every Lie triple system, with triple product denoted $[abc]$ is a subspace of some Lie algebra, with product denoted $[a, b]$, such that $[abc] = [[a, b], c]$.
(Meyberg Lectures, chapter 6, Theorem 1, page 45)

28. Find out what a semisimple associative algebra is and prove that every derivation of a finite dimensional semisimple associative algebra is inner, that is, of the form $x \mapsto ax - xa$ for some fixed a in the algebra.
(G. Hochschild, Semisimple algebras and generalized derivations, Amer. J. Math. 64, 1942, 677–694, Theorem 2.2)

29. Find out what a semisimple Lie algebra is and prove that every derivation of a finite dimensional semisimple Lie algebra is inner, that is, of the form $x \mapsto [a, x]$ for some fixed a in the algebra.
(Meyberg Lectures, chapter 5, Theorem 2, page 42; see also G. Hochschild, Semisimple algebras and generalized derivations, Amer. J. Math. 64, 1942, 677–694, Theorem 2.1)

30. Find out what a semisimple Jordan algebra is and prove that every derivation of a finite dimensional semisimple Jordan algebra is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}(a_i(b_i x) - b_i(a_i x))$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the algebra.
(N. Jacobson, Structure of Jordan algebras, 1968, around page 320 and Braun-Koecher, Jordan Algebren, around page 270)

31. In an associative triple system with triple product $\langle xyz \rangle$, show that you get a Jordan triple system with the triple product $\{xyz\} = \langle xyz \rangle + \langle zyx \rangle$. Then use Theorem 7 to prove Theorem 5.

    THEOREM 7
    EVERY DERIVATION OF $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

    THEOREM 5
    EVERY DERIVATION ON $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

32. Find out what a semisimple associative triple system is and prove that every derivation of a finite dimensional semisimple associative triple system is inner (also find out what inner means in this context).
(R. Carlsson, Cohomology for associative triple systems, P.A.M.S. 1976, 1–7)

33. Find out what a semisimple Lie triple system is and prove that every derivation of a finite dimensional semisimple Lie triple system is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}[a_i b_i x]$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the Lie triple system.
(Meyberg Lectures, chapter 6, Theorem 10, page 57)

34. Find out what a semisimple Jordan triple system is and prove that every derivation of a finite dimensional semisimple Jordan triple system is inner, that is, of the form $x \mapsto \sum_{i=1}^{n}(\{a_i b_i x\} - \{b_i a_i x\})$ for some fixed elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in the Jordan triple system.
(Meyberg Lectures, chapter 11, Theorem 8, page 123 and Corollary 2, page 124)