Elementary Analysis Math 140B—Winter 2007
Homework answers—Assignment 1; January 17, 2007

Exercise 23.4, page 176
For $n = 0, 1, \ldots$, let $a_n = \left[\frac{4+2(-1)^n}{5}\right]^n$

(a) Find $\limsup a_n^{1/n}$, $\liminf a_n^{1/n}$, $\limsup \left|\frac{a_{n+1}}{a_n}\right|$ and $\liminf \left|\frac{a_{n+1}}{a_n}\right|$.

Solution:

- $\{a_n^{1/n}\} = \{\frac{6}{5}, \frac{2}{5}, \frac{6}{5}, \frac{2}{5}, \ldots\}$, so $\limsup a_n^{1/n} = 6/5$ and $\liminf a_n^{1/n} = 2/5$.
- $\{\frac{a_{n+1}}{a_n}\} = \{\frac{2}{1}, \frac{6^2}{2}, \frac{2^3}{6^2}, \frac{6^4}{2^3}, \ldots\}$, and $\lim \frac{2^{k+1}}{6^{2k}} = 0$ and $\lim \frac{6^{2k}}{2^{2k-1}} = +\infty$. Thus $\limsup \left|\frac{a_{n+1}}{a_n}\right| = +\infty$ and $\liminf \left|\frac{a_{n+1}}{a_n}\right| = 0$.

(b) Do the series $\sum a_n$ and $\sum(-1)^n a_n$ converge?

Solution:

- The series $\sum a_n$ diverges by the root test (Theorem 14.9, page 94). Alternatively, it diverges since the general term does not converge to 0: $\{a_n\} = \{1, \frac{2}{5}, (\frac{6}{5})^2, (\frac{2}{5})^3, (\frac{6}{5})^4, (\frac{2}{5})^5, (\frac{6}{5})^6, \ldots\}$
- The series $\sum(-1)^n a_n$ diverges for exactly the same reasons as $\sum a_n$ does.

(c) Find the radius of convergence and determine the interval of convergence for the series $\sum a_n x^n$.

Solution:

- $\beta = \limsup |a_n|^{1/n} = 6/5$, so $R = 1/\beta = 5/6$ is the radius of convergence.
- If $x = 6/5$, we have $\sum a_n \left(\frac{6}{5}\right)^n = 1 + \frac{2}{5} \cdot \frac{5}{6} + \left(\frac{6}{5}\right)^2 \left(\frac{5}{6}\right)^2 + \left(\frac{2}{5}\right)^3 \left(\frac{5}{6}\right)^3 + \cdots = 1 + \frac{2}{6} + 1 + \left(\frac{2}{6}\right)^3 + 1 + \cdots$ diverges.
- If $x = -6/5$, we have $\sum a_n \left(-\frac{6}{5}\right)^n = 1 - \frac{2}{6} + 1 - \left(\frac{2}{6}\right)^3 + 1 + \cdots$ diverges.
- So the interval of convergence is the open interval $(-5/6, 5/6)$.

Exercise 23.5, page 176
Consider a power series $\sum a_n x^n$ with radius of convergence R.

(a) Prove that if all the coefficients a_n are integers and if infinitely many are non zero, then $R \le 1$

Solution: There is a subsequence a_{n_k} such that $|a_{n_k}| \ge 1$, so that $|a_{n_k}|^{1/n_k} \ge 1$ and $\beta = \limsup |a_n|^{1/n} \ge \limsup_k |a_{n_k}|^{1/n_k} \ge 1$. Hence $R = 1/\beta \le 1$.

(b) Prove that if $\limsup |a_n| > 0$, then $R \le 1$.

Solution: Let c be any number between 0 and $\limsup |a_n|$. There is a subsequence $|a_{n_k}|$ converging to $\limsup |a_n|$ and so $|a_{n_k}| > c$ for all sufficiently large k. Then $|a_{n_k}|^{1/n_k} > c^{1/n_k}$ and $\limsup_k |a_{n_k}|^{1/n_k} \ge \lim_k c^{1/n_k} = 1$. Hence $\beta = \limsup |a_n|^{1/n} \ge \limsup_k |a_{n_k}|^{1/n_k} \ge 1$ and $R = 1/\beta \le 1$.