1. (a) The line segment from 4−6i to 7−5i; the Cartesian equation is \( x - 4 = 3y + 18 \)
(b) The line segment from \( w \) to \( z \); if you insist, the equation would be
\[
\frac{x - \Re w}{\Re z - \Re w} = \frac{y - \Im w}{\Im z - \Im w}
\]
where \( w = \Re w + i\Im w \), \( z = \Re z + i\Im z \).

2. \( \gamma(t) = t + i(4t^3 - 1), 0 \leq t \leq 10. \)

3. \( \gamma(t) = 2 \cos t + 3 + i(2 \sin t - 1), 0 \leq t \leq 2\pi. \)

4. bye

5. \( f \) is continuous, so \( \lim_{z \to 3 + 2i} f(z) = f(3 + 2i) = 18 - i \)

6. \( \lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0) = f'(z_0) \cdot 0 = 0. \)

7. \( f(z) - f(z_0) = \overline{z} - \overline{z_0} = (x - x_0) - i(y - y_0), \) where \( z = x + iy \) and \( z_0 = x_0 + iy_0. \)
First let \( z \) approach \( z_0 \) horizontally, so that \( z = x + iy_0. \) Then
\[
\frac{f(z) - f(z_0)}{z - z_0} = \frac{x - x_0}{x - x_0} = 1
\]
so that if \( f'(z_0) \) existed, it would equal 1. On the other hand, if \( z \) approaches \( z_0 \) vertically, so that \( z = x_0 + iy \), then
\[
\frac{f(z) - f(z_0)}{z - z_0} = \frac{-i(y - y_0)}{i(y - y_0)} = -1
\]
so that if \( f'(z_0) \) existed, it would equal -1. Thus \( f'(z_0) \) does not exist at any point \( z_0. \)

8. This is a polynomial in \( z \) and so differentiable everywhere, and \( f'(z) = 3(2 + i)z^2 - 2iz + 4. \)

9. NO.
\[
\frac{f(z) - f(0)}{z - 0} = \frac{(z)^2}{z^2} = \frac{(x - iy)^2}{(x + iy)^2} = \frac{x^2 - y^2 - 2ixy}{x^2 - y^2 + 2ixy}.
\]
Along the line \( y = x \), this approaches \(-1\), and along the line \( y = 0 \), this approaches 1. So \( f'(0) \) does not exist.

10. The Cauchy-Riemann equations are satisfied at \( z = i \) and nowhere else. The partial derivatives are continuous everywhere, and in particular at \( z = i \), so \( f \) is differentiable at \( z = i \) only, and hence analytic nowhere.
11. YES, the Cauchy-Riemann equations are satisfied at \((0,0)\). However, with \(f = u + iv\), \(u_x\) and \(v_y\) are not continuous at \((0,0)\). This explains why \(f'(0)\) does not exist. Here are some details:

We have \(u(x, y) = (x^3 - 3xy^2)/(x^2 + y^2)\) for \((x, y) \neq (0,0)\), and \(u(0,0) = 0\), and \(v(x, y) = (y^3 - 3xy^2)/(x^2 + y^2)\) for \((x, y) \neq (0,0)\), and \(v(0,0) = 0\). So
\[u_x(0,0) = \lim_{x \to 0} (u(x,0) - u(0,0))/x = \lim_{x \to 0} x/x = 1\]
and
\[v_y(0,0) = \lim_{y \to 0} (v(0,y) - v(0,0))/y = \lim_{x \to 0} y/y = 1.\]
This shows that one of the two Cauchy-Riemann equations is satisfied at \((0,0)\).

However, for \((x, y) \neq (0,0)\), \(u_x(x, y) = (x^4 - 3y^4 + 6x^2y^2)/(x^2+y^2)^2\). If \((x,y) \to (0,0)\) along the line \(y = x\) then \(u_x(x,x) = 4x^4/4x^4 = 1\); if \((x,y) \to (0,0)\) along the line \(x = 0\) then \(u_x(0,y) = -3y^4/y^4 = -3\). Therefore \(\lim_{(x,y) \to (0,0)} u_x(x,y)\) does not exist, and so \(u_x\) is not continuous at \((0,0)\).

12. NOWHERE.

13. If \(f = u + iv\), then \(f'(z) = u_x(x,y) + iv_x(x,y)\) and therefore \(u_x = v_y = v_x = u_y = 0\) in \(D\). By the mean value theorem, for any two points \(z_1, z_2\) in \(D\) which can be connected by a straight line lying in \(D\), we can find points \((x_3,y_3)\) and \((x_4,y_4)\) on that straight line such that
\[
f(z_2) - f(z_1) = u(x_2,y_2) - u(x_1,y_1) + i(v(x_2,y_2) - v(x_1,y_1))
= u_x(x_3,y_3)(x_2-x_1) + u_y(x_3,y_3)(y_2-y_1) + i[v_x(x_4,y_4)(x_2-x_1) + v_y(x_4,y_4)(y_2-y_1)].
\]
Therefore \(f(z_1) = f(z_2)\). Since \(D\) is connected, any two points \(z,w\) in \(D\) can be connected by a finite sequence of line segments lying in \(D\), and it follows that \(f(z) = f(w)\) and \(f\) is a constant.

14. The function \(f = u + iv\) is defined on \(C - \{0\}\). The Cauchy-Riemann equations are satisfied everywhere on \(C - \{0\}\) and the partial derivatives \(u_x\) and \(u_y\) are continuous on \(C - \{0\}\). Therefore \(f\) is differentiable at every point of \(C - \{0\}\), and therefore analytic on \(C - \{0\}\).

15. NO. Let \(f(z) = 1\) if \(|z| < 1\) and \(f(z) = 1+i\) on \(\{z:|z-3|<1\}\). So \(f\) is defined and non constant on the open set \(D = \{z:|z|<1\} \cup \{z:|z-3|<1\}\), but \(\Re f\) is constant on \(D\).

If you add the condition that \(D\) is connected, then from \(u_x = u_y = 0\) the Cauchy-Riemann equations tell you that \(v_x = v_y = 0\) and by Problem 13, \(f\) is a constant.

16. NO. Let \(f(z) = 1\) if \(|z| < 1\) and \(f(z) = i\) on \(\{z:|z-3|<1\}\).

However, again, if \(D\) is connected then \(f\) is a constant.
PROOF: $u^2 + v^2 = c$ implies that

$$uu_x + vv_x = 0 = uu_y + vv_y$$

Using the Cauchy-Riemann equations, this becomes

$$uv_y - vu_y = 0 = vv_y + uu_y$$

or in matrix form

$$\begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} v_y \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 

The determinant of this matrix is $u^2 + v^2 = c$, so if $c = 0$ then $u = v = 0$ so $f = 0$ is constant. If $c \neq 0$, then by linear algebra, $v_y = u_y = 0$ and we can use Problem 13 again.