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§2 $\mathfrak{g}$ is a Leibniz algebra $\quad ad(X)Y := [X,Y]_{\mathfrak{g}}$

Jacobi identity : $ad X \in Der(\mathfrak{g})$ $\quad [x,[y,z]] = [[x,y],z] + [y,[x,z]]$ (sprinkle with subscript $\mathfrak{g}$)

$\mathcal{S} :=$ the ideal generated by $\{[x,x]_{\mathfrak{g}} : x \in \mathfrak{g}\} = \operatorname{span}\{[x,x]_{\mathfrak{g}} : x \in \mathfrak{g}\}$

$\mathfrak{h} := \mathfrak{g}/\mathcal{S}$ is a Lie algebra $\quad \bar{x} = x + \mathcal{S}, \quad [\bar{x},\bar{y}]_{\mathfrak{h}} := [x,y]_{\mathfrak{g}} + \mathcal{S}$

$\pi : \mathfrak{g} \to \mathfrak{h} = \mathfrak{g}/\mathcal{S} \quad \pi(x) = \bar{x}$ is a homomorphism of Leibniz algebras

$$\pi([x,y]_{\mathfrak{g}}) \overset{?}{=} [\pi(x),\pi(y)]_{\mathfrak{h}}$$
$$\pi([x,y]_{\mathfrak{g}}) = \overline{[x,y]_{\mathfrak{g}}}, \quad [\pi(x),\pi(y)]_{\mathfrak{h}} = [\bar{x},\bar{y}]_{\mathfrak{h}} = \overline{[x,y]_{\mathfrak{g}}}$$

π is also a $\boxed{\text{morphism}}$ of left $\mathfrak{h}$-modules !!

$\mathfrak{g}$ is an left $\mathfrak{h}$-module via $\quad \mathfrak{h} \times \mathfrak{g} \to \mathfrak{g}, \quad (\bar{x},y) \to [x,y]_{\mathfrak{g}} = \bar{x} \cdot y$

$\mathfrak{h}$ is an left $\mathfrak{h}$-module via $\quad \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}, \quad (\bar{x},\bar{y}) \to \overline{[x,y]_{\mathfrak{g}}} = \bar{x} \cdot \bar{y}$

$\pi : \mathfrak{g} \to \mathfrak{h}, \quad \pi(x) = \bar{x} = x + \mathcal{S}, \quad \boxed{\pi(\bar{x} \cdot y) \overset{?}{=} \bar{x} \cdot \pi(y)}$

$$\pi(\bar{x} \cdot y) = \pi([x,y]_{\mathfrak{g}}) = \overline{[x,y]_{\mathfrak{g}}}, \qquad \bar{x} \cdot \pi(y) = \bar{x} \cdot \bar{y} = \overline{[x,y]_{\mathfrak{g}}}$$

Elaboration of the module definitions

$\mathfrak{h}$ is a Lie algebra (hence a Leibniz algebra) so $\mathfrak{g}$ being an $\mathfrak{h}$-module (more precisely a Leibniz $\mathfrak{h}$-module) also means

$$\underset{\in \mathfrak{h}}{[\bar{x}_1,\bar{x}_2]_{\mathfrak{h}}} \cdot \underset{\in \mathfrak{g}}{y} = \bar{x}_1 \cdot (\bar{x}_2 \cdot y) - \bar{x}_2 \cdot (\bar{x}_1 \cdot y) = [x_1,[x_2,y]_{\mathfrak{g}}]_{\mathfrak{g}} - [x_2,[x_1,y]_{\mathfrak{g}}]_{\mathfrak{g}}$$

$\mathfrak{h}$ is a Leibniz algebra so being an $\mathfrak{h}$-module also means

$$\underset{\in \mathfrak{h}}{[\bar{x}_1,\bar{x}_2]_{\mathfrak{h}}} \cdot \underset{\mathfrak{h}}{\bar{y}} = \bar{x}_1 \cdot (\bar{x}_2 \cdot \bar{y}) - \bar{x}_2 \cdot (\bar{x}_1 \cdot \bar{y}) = [\bar{x}_1, \overline{[x_2,y]_{\mathfrak{g}}}]_{\mathfrak{g}} - [\bar{x}_2, \overline{[x_1,y]_{\mathfrak{g}}}]_{\mathfrak{g}}$$

$$= \overline{[x_1,[x_2,y]_{\mathfrak{g}}]_{\mathfrak{g}} - [x_2,[x_1,y]_{\mathfrak{g}}]_{\mathfrak{g}}}$$

$$= \overline{[x_1,[x_2,y]_{\mathfrak{g}}]_{\mathfrak{g}}} - \overline{[x_2,[x_1,y]_{\mathfrak{g}}]_{\mathfrak{g}}}$$

LHS $= \overline{[x_1,x_2]_{\mathfrak{g}}} \cdot \bar{y} = \overline{[[x_1,x_2]_{\mathfrak{g}},y]_{\mathfrak{g}}}$

The Leibniz identity

equal !

this also ~~#~~ $= [x_1,x_2]_{\mathfrak{g}} \cdot y$

$= [[x_1,x_2]_{\mathfrak{g}},y]_{\mathfrak{g}}$

$\mathfrak{g}$ Leibniz alg $\quad \mathcal{S} = \mathrm{span}\{[x,x]_{\mathfrak{g}} : x \in \mathfrak{g}\}$ ideal, $\bar{x} = x + \mathcal{S}$ (coset)

$\mathfrak{h} := \mathfrak{g}/\mathcal{S}$ is Lie algebra $\quad [\bar{x},\bar{y}]_{\mathfrak{h}} = \overline{[x,y]_{\mathfrak{g}}}$

$\mathfrak{g}$ is a left $\mathfrak{h}$-module means $\exists$ bilinear map $\mathfrak{h} \times \mathfrak{g} \to \mathfrak{g}$

$$(\bar{x}, y) \to \bar{x} \circ y$$

which satisfies, besides $(\bar{x}_1 + \bar{x}_2) \circ y = \bar{x}_1 \circ y + \bar{x}_2 \circ y$

and $\bar{x} \circ (y_1 + y_2) = \bar{x} \circ y_1 + \bar{x} \circ y_2$

also $[\bar{x}_1, \bar{x}_2]_{\mathfrak{h}} \circ y = \bar{x}_1 \circ (\bar{x}_2 \circ y) - \bar{x}_2 \circ (\bar{x}_1 \circ y)$ ← this is proved on the previous page

The module action is $(\bar{x}, y) \to \bar{x} \circ y = [x,y]_{\mathfrak{g}}$

$\mathfrak{h}$ is a left $\mathfrak{h}$-module means $\exists$ bilinear map $\mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$

$$(\bar{x}, \bar{y}) \to \bar{x} \circ \bar{y}$$

which satisfies, besides $(\bar{x}_1 + \bar{x}_2) \circ \bar{y} = \bar{x}_1 \circ \bar{y} + \bar{x}_2 \circ \bar{y}$

and $\bar{x} \circ (\bar{y}_1 + \bar{y}_2) = \bar{x} \circ \bar{y}_1 + \bar{x} \circ \bar{y}_2$

also $[\bar{x}_1, \bar{x}_2]_{\mathfrak{h}} \circ \bar{y} = \bar{x}_1 \circ (\bar{x}_2 \circ \bar{y}) - \bar{x}_2 \circ (\bar{x}_1 \circ \bar{y})$

↑ this is proved on the previous page

In this case, since $\mathfrak{h}$ is a Lie algebra (hence a Leibniz algebra) we take the module action to be the Lie product in $\mathfrak{h}$, i.e.

$$(\bar{x}, \bar{y}) \to \bar{x} \circ \bar{y} = [\bar{x}, \bar{y}]_{\mathfrak{h}} = \overline{[x,y]_{\mathfrak{g}}}$$

$\pi : \mathfrak{g} \to \mathfrak{h} = \mathfrak{g}/\mathcal{S} \quad \pi(x) = \bar{x}$ is a morphism of Leibniz algebras

means π is linear and $\pi([x,y]_{\mathfrak{g}}) = [\pi(x), \pi(y)]_{\mathfrak{h}}$

↑ this is proved on the previous page

$\pi : \mathfrak{g} \to \mathfrak{h} = \mathfrak{g}/\mathcal{S} \quad \pi(x) = \bar{x}$ is a morphism of left $\mathfrak{h}$-modules

means π is linear and $\pi(\bar{x} \circ y) = \bar{x} \circ \pi(y)$

(i.e. $\mathfrak{h}$ acts like a scalar) proof

$$\pi(\bar{x} \circ y) = \pi([x,y]_{\mathfrak{g}}) = \overline{[x,y]_{\mathfrak{g}}} = [\bar{x}, \bar{y}]_{\mathfrak{h}} = \bar{x} \circ \bar{y} = \bar{x} \circ \pi(y)$$

Def $\mathfrak{g}$ Leibniz alg, $\mathcal{E} \subset \mathfrak{g}$ ideal, $\mathcal{S} \subset \mathcal{E} \subset \ker(\mathrm{ad}) \subset \mathfrak{g}$

$\mathfrak{g}$ splits over $\mathcal{E}$ if $\exists$ Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$

$\mathfrak{g} = \mathcal{E} \oplus \mathfrak{h}$ (direct sum of vector spaces)

In this case $[u+X, v+Y] = X\cdot v + [X,Y]$

for $u, v \in \mathcal{E}$, $X, Y \in \mathfrak{h}$ $\qquad [X,Y] = [X,Y]_{\mathfrak{h}} = [X,Y]_{\mathfrak{g}}$

$$[u+X, v+Y] = \underbrace{[u,v]_{\mathfrak{g}} + [u,Y]_{\mathfrak{g}}}_{=0 \text{ since } \mathrm{ad}\, u = 0} + [X,v]_{\mathfrak{g}} + [X,Y]_{\mathfrak{h}}$$

$[X,v]_{\mathfrak{g}} = X\cdot v$ since $\mathcal{E}$ is an $\mathfrak{h}$-module

since $[X,Y]_{\mathfrak{h}}\cdot v = [[X,Y]_{\mathfrak{h}}, v]_{\mathfrak{g}} = [X,[Y,v]_{\mathfrak{g}}]_{\mathfrak{g}} - [Y,[X,v]_{\mathfrak{g}}]_{\mathfrak{g}}$

$= X\cdot(Y\cdot v) - Y\cdot(X\cdot v)$

Theorem 2.1 Let $\mathfrak{g}$ be a Leibniz algebra, $\mathcal{E}$ an ideal of $\mathfrak{g}$ with $\mathcal{S} \subset \mathcal{E} \subset \ker(\mathrm{ad})$. Then $\mathfrak{g}$ splits over $\mathcal{E}$ (i.e. $\exists$ Lie subalg $\mathfrak{h} \subset \mathfrak{g}$ with $\mathfrak{g} = \mathcal{E} \oplus \mathfrak{h}$ as vector spaces) if and only if $\mathfrak{g}$ is a demisemidirect product of $\mathcal{E}$ with a Lie algebra $\mathfrak{h} \in \mathfrak{g}$ (ie. $\mathfrak{g} = V \oplus \mathfrak{h}$ with the bracket $[u+X, v+Y]_{\mathfrak{g}} := X\cdot v + [X,Y]_{\mathfrak{h}}$) ($V$ is an $\mathfrak{h}$-module)

If $\mathfrak{g}$ splits, it is a demisemidirect product by the top of this page.

Suppose ~~first~~ next that $\mathfrak{g} = V \oplus \mathfrak{h}$ is a demisemidirect product

Need to prove $\underbrace{[u+X,[v+Y, w+Z]]}_{LHS} = \underbrace{[v+Y,[u+X, w+Z]]}_{RHS_1} + \underbrace{[[u+X, v+Y], w+Z]}_{RHS_2}$

$LHS = [u+X, Y\cdot w + [Y,Z]] = X\cdot(Y\cdot w) + [X,[Y,Z]]$

$RHS_2 = [X\cdot v + [X,Y], w+Z] = [X,Y]\cdot w + [[X,Y],Z]$

$RHS_1 = [v+Y, X\cdot w + [X,Z]] = Y\cdot(X\cdot w) + [Y,[X,Z]]$

$LHS = RHS_1 + RHS_2$

$\mathfrak{g}$ is a Leibniz algebra which splits $\square$

<u>Remarks</u> 1. If Leibniz alg $\mathfrak{g}$ splits over $\mathcal{S}$ (i.e. $\mathcal{E}=\mathcal{S}$) with complementary Lie algebra $\mathfrak{h}$, then $\mathfrak{h}\cdot\mathcal{S}=\mathcal{S}$

($\mathfrak{h}\cdot\mathcal{S}$ means span$\{X\cdot u : X\in\mathfrak{h}, u\in\mathcal{S}\}$).

$\mathfrak{g}=\mathcal{S}\oplus\mathfrak{h}$ so if $X\in\mathfrak{h}$, $u\in\mathcal{S}$, then

$X\cdot u=[X,u]_{\mathfrak{g}}\in\mathcal{S}$ since $\mathcal{S}$ is an ideal.

$\therefore\ \mathfrak{h}\cdot\mathcal{S}\subset\mathcal{S}$

Conversely, if $u\in\mathcal{S}$, $u=\sum\lambda_i\,[u_i+X_i,u_i+X_i]_{\mathfrak{g}}$ $\quad u_i\in\mathcal{S}$, $X_i\in\mathfrak{h}$

so $u=\sum\big(\lambda_i(X_i\cdot u_i)+\underbrace{[X_i,X_i]_{\mathfrak{h}}}_{=0}\big)=\sum\lambda_i(X_i\cdot u_i)\in\mathfrak{h}\cdot\mathcal{S}$

2. Conversely if $\mathfrak{g}=V\oplus\mathfrak{h}$ is a demisemidirect product so V is an $\mathfrak{h}$ module & $\mathfrak{h}$ is a Lie (sub)algebra and if $\mathfrak{h}\cdot V=V$, then $\mathcal{S}\simeq V$.

proof ?

Example 2.2

Example 2.3