Kinyon JLieTh 2007 kinyonTHM3.4.pdf 12/4/17 I.

continued 4/24/18 and 5/11/18

§ 3 Lie racks (and § 4,5)

def 3.1 pointed rack $(Q, \circ, 1)$

- Q is a set
- $\circ : Q \times Q \to Q$ binary operation
- 1 distinguished element

1. $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$ (left distributivity)
2. $\forall a, b \in Q \ \exists ! x \in Q, \ a \circ x = b$
3. $1 \circ x = x \quad x \circ 1 = 1 \quad \forall x \in Q$

$(Q, \circ)$ any **magma** (i.e. a set with a binary operation)

$$\mathrm{Aut}\, Q = \{ \psi \text{ bijection} : Q \to Q, \ \psi(x \circ y) = \psi(x) \circ \psi(y) \}$$

If : $\phi(x) y := x \circ y$ (left translation), then $\phi(x) \in \mathrm{Aut}\, Q$.

Example: G a group $x \circ y = x y x^{-1}$ $(G, \circ, 1)$ is pointed rack.

survey of "racks"

[1] Andruskiewitsch, N & Graña, M Adv Math 178 '03 177–243

def 3.2 Lie rack $(Q, \circ, 1)$ = smooth manifold with $\circ : Q \times Q \to Q$ is smooth.

Examples • conjugation in a Lie group

Example 3.3 H Lie gp, V H-module $Q := V \times H$

$$(u, A) \circ (v, B) = (Av, ABA^{-1}) \qquad u, v \in V, \ A, B \in H$$

$1 = (0, 1)$ $(Q, \circ, 1)$ is a linear Lie rack. (by definition)

Example 3.3 details

V being an H-module (meaning H-Lie group module)

means $\exists$ $\quad H \times V \longrightarrow V \quad$ (V a vector space)

$$(A, u) \longrightarrow A \circ u$$

smooth

such that $\quad 1 \circ u = u \quad$ (1 = identity element of H)

$$A \circ (\alpha u + v) = \alpha A \circ u + A \circ v$$

$$(AB) \circ u = A \circ (B \circ u)$$

To verify that $\quad Q := V \times H, \quad 1 = (0, 1)$ (zero vector in V; identity elt of H)

and $\quad (u, A) \circ (v, B) := (Av, ABA^{-1})$

is a Lie rack, need to prove

1. $(u, A) \circ ((v, B) \circ (w, C)) = ((u, A) \circ (v, B)) \circ ((u, A) \circ (w, C))$
2. given $(u, A), (v, B)$, $\exists!$ (x, X) with
$$(u, A) \circ (x, X) = (v, B)$$
3. $1 \circ (u, A) = (u, A) \quad$ and $\quad (u, A) \circ 1 = 1$

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1. $\text{LHS} = (u, A) \circ (B \circ w, BCB^{-1}) = (A \circ (B \circ w), ABCB^{-1}A^{-1})$

   $\text{RHS} = (Av, ABA^{-1}) \circ (Aw, ACA^{-1}) = ((ABA^{-1}) \circ (Aw), ABA^{-1}ACA^{-1}AB^{-1}A^{-1})$
   $= ((AB) \circ w, ABCB^{-1}A^{-1})$

2. $\text{LHS} = (A \circ x, AXA^{-1})$

   need $A \circ x = v$ and $AXA^{-1} = B$

   so let $x = A^{-1} \circ v$, $X = A^{-1}BA$ proving existence of (x, X)

   For uniqueness, if (y, Y) also satisfies $A \circ y = v$, $AYA^{-1} = B$

   then $y = A^{-1} \circ (A \circ y) = A^{-1} \circ (A \circ x) = x$ and $Y = A^{-1}BA = X$

3. $1 \circ (u, A) = (0, 1) \circ (u, A) = (u, 1A1^{-1}) = (u, A)$

   $(u, A) \circ 1 = (u, A) \circ (0, 1) = (0, A1A^{-1}) = (0, 1) = \underline{1}$

non-pointed smooth racks

[20] Sabinin, L.V. Smooth Quasigroups & Loops (book 1999)

Theorem 3.4 $(Q, \circ, 1)$ a Lie rack $\mathfrak{g} = T_1 Q \Rightarrow$

$\exists$ bilinear $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

(1) $(\mathfrak{g}, [\cdot,\cdot])$ is a Leibniz algebra (the tangent Leibniz algebra of the rack)

(2) $\forall x \in Q$, the tangent map $\Phi(x) = T_1 \phi(x) : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of $(\mathfrak{g}, [\cdot,\cdot])$

(3) $T_1 \Phi = \mathrm{ad}$ where $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

$\mathrm{ad}\, X\,(Y) = [X, Y]$.

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proof For $x \in Q$, $\phi(x) 1 = 1$ so $\phi(x) : Q \to Q$ satisfies $\phi(x)'(z) : T_1 Q \to T_1 Q$ for $z \in Q$.

$$\boxed{\begin{array}{l}\text{recall if } f : Q \to Q \text{ and } p \in Q, \text{ then } f'(p) \text{ is a} \\ \text{linear transformation from } T_p Q \text{ to } T_{f(p)} Q \\ \text{defined by } f'(p) \cdot \xi = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) \quad \text{where } \gamma : \mathbb{R} \to Q \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \gamma(0) = p \quad \gamma'(0) = \xi\end{array}}$$

We define $\Phi(x) = T_1 \phi(x)$ to be $\phi(x)'(1) : T_1 Q \to T_1 Q$

Then $\Phi : Q \to L(T_1 Q)$ (= all linear transf on the vector space $T_1 Q$)

$\Phi(x) = \phi(x)'(1)$. Since $\phi(x)$ is a bijection, its derivative at any point is an invertible linear transformation

If $f \circ f^{-1} = \mathrm{id}_Q$ then $f'(f^{-1}(z)) \circ f^{-1\,\prime}(z) = \mathrm{id}_Q{}'(z)$

$$\mathrm{id}_Q{}'(z) \cdot \xi = \frac{d}{dt}\, \mathrm{id}_Q(\gamma(t)) \qquad \gamma(0) = z,\ \gamma'(0) = \xi$$

$$= \lim_{t \to 0} \frac{\mathrm{id}_Q(\gamma(t)) - \mathrm{id}_Q(z)}{t} = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t} = \gamma'(0) = \xi$$

So $\mathrm{id}_Q{}'(z) = \mathrm{id}_{T_zQ}$ so $f^{-1\,\prime}(z)$ has an inverse

and the same holds for $f'(z)$.

Since $\Phi(x)$ is invertible as an element of $L(T_1Q)$ we may write $\Phi : Q \longrightarrow GL(T_1Q)$

Moreover $\Phi(1) = \mathrm{id}_{T_1Q}$ since $\phi(1) = \mathrm{id}_Q$.

Hence $\Phi'(z) : T_zQ \longrightarrow T_{\Phi(z)}(GL(T_1Q))$

and $\Phi'(1) : T_1Q \to T_{\mathrm{id}_{T_1Q}}(GL(T_1Q)) = gl(T_1Q)$ $(= L(T_1Q))$

We define $\mathrm{ad} : T_1Q \to gl(T_1Q)$ by $\mathrm{ad} = \Phi'(1)$

and set $[\xi, \eta] := \mathrm{ad}(\xi)(\eta)$ $(\xi, \eta \in T_1Q)$

It remains to prove

$$\Phi(x)[\xi, \eta] = [\Phi(x)\xi, \Phi(x)\eta]$$ (which is (2) in the theorem)

and

$$[\xi, [\eta, \zeta]] = [[\xi, \eta], \zeta] + [\eta, [\xi, \zeta]]$$

((3) holds by definition) (which is (1) in the theorem)