continues kinyonTHM3.4.pdf

Here is the proof of

$$(*) \quad \Phi(x)[\xi,\eta] = [\Phi(x)\xi, \Phi(x)\eta] \qquad \xi,\eta \in T_1Q$$

Recall that $(Q, \circ, 1)$ is a Lie rack

$$\phi(x)y = x \circ y \qquad \phi(x): Q \to Q$$

$$\Phi(x) = \phi(x)'(1) : T_1Q \to T_1Q$$

The identity $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$ can be stated as

$$\phi(x)\phi(y) = \phi(\phi(x)y)\phi(x)$$

We let $f = \phi(x)\phi(y)$ so $f(z) = \phi(x)\phi(y)z = x \circ (y \circ z)$

and $g = \phi(\phi(x)y)\phi(x)$ so $g(z) = \phi(\phi(x)y)\phi(x)z = (x \circ y) \circ (x \circ z)$

We calculate $f'(z)$ where the derivative is with respect to z.

$$f'(z) = \phi(x)'(\phi(y)z) \circ \phi(y)'(z) \quad : T_1Q \to T_1Q$$

$$f'(1) = \phi(x)'(1) \circ \phi(y)'(1) = \Phi(x) \circ \Phi(y)$$

Now let $h(y) = f'(1) = \Phi(x) \circ \Phi(y) \qquad h: Q \to L(T_1Q)$

Then $h'(1) = \Phi(x) \circ \Phi'(1) = \Phi(x) \circ ad \qquad h'(1): T_1Q \to L(T_1Q)$

(recall that $ad = \Phi'(1)$ where $\Phi: Q \to L(T_1Q)$)

$h'(1)(\xi).\eta = \Phi(x) \circ ad\,\xi\,(\eta) = \Phi(x)([\xi,\eta])$

Now we calculate $g'(z)$ (recall that $f=g$ so $f'(z)=g'(z)$)

$$g'(z) = \Phi(\phi(x)y)'(\phi(x)z) \circ \phi(x)'(z)$$

$$g'(1) = \Phi(\phi(x)y)'(1) \circ \Phi(x) = \Phi(\phi(x)y) \circ \Phi(x)$$

Then $h(y) = f'(1) = g'(1) = \Phi(\phi(x)y) \circ \Phi(x) \quad : T_1Q \to T_1Q$

To calculate $h'(y)$ consider

$$Q \xrightarrow{\phi(x)} Q \xrightarrow{\Phi} L(T_1Q) \xrightarrow{R_{\Phi(x)}} L(T_1Q)$$

$$y \longrightarrow \phi(x)y \longrightarrow \Phi(\phi(x)y) \longrightarrow \Phi(\phi(x)y) \circ \Phi(x)$$

(where $R_S(T) = T \circ S \quad R_S : L(T_1Q) \to L(T_1Q)$)

So $h(y) = R_{\Phi(x)} \circ \Phi \circ \phi(x)\ (y)$

R_S is a linear transf $T_1Q \to T_1Q$ so $R_S'(T) = R_S \ \forall T$.

i.e. $h = R_{\Phi(x)} \circ \Phi \circ \phi(x)$

so $h'(y) = R_{\Phi(x)}'(\Phi(\phi(x)y)) \circ \Phi'(\phi(x)y) \circ \phi(x)'(y)$

and $h'(1) =$

$$R_{\Phi(x)} \circ \Phi'(1) \circ \Phi(x)$$

$$h'(1)(\xi) = R_{\Phi(x)}(\Phi'(1)(\Phi(x)\xi)) = R_{\Phi(x)}(\mathrm{ad}(\Phi(x)\xi))$$

$$= \mathrm{ad}(\Phi(x)\xi) \circ \Phi(x)$$

$$h'(1)(\xi).\eta = \mathrm{ad}(\Phi(x)\xi)(\Phi(x)\eta) = [\Phi(x)\xi, \Phi(x)\eta]$$ proving (*)

Here is the proof of

(**) $[\xi,[\eta,\zeta]] = [[\xi,\eta],\zeta] + [\eta,[\xi,\zeta]]$

We know (*) $\Phi(x)[\xi,\eta] = [\Phi(x)\xi, \Phi(x)\eta]$

let $f(x) = [\Phi(x)\xi, \Phi(x)\eta]$ and $g(x) = \Phi(x)[\xi,\eta]$

Then $g'(x) = \Phi'(x)([\xi,\eta])$

so $g'(1)(\zeta) = \Phi'(1)([\xi,\eta]).\zeta = ad([\xi,\eta]).\zeta = [[\xi,\eta],\zeta]$

Now we calculate $f'(1)$. We can write $f = B\circ P$

$$Q \xrightarrow{P} T_1Q \times T_1Q \xrightarrow{B} T_1Q$$

$$x \longrightarrow (\Phi(x)\xi, \Phi(x)\eta) \longrightarrow [\Phi(x)\xi, \Phi(x)\eta]$$

where $P: Q \to T_1Q \times T_1Q$ is $P(x) = (\Phi(x)\xi, \Phi(x)\eta)$ (cartesian product)

and $B: T_1Q \times T_1Q \to T_1Q$ is $B(\alpha,\beta) = [\alpha,\beta]$ a bilinear map
$= ad\alpha(\beta)$

So $f = B\circ P$ $\quad f'(x) = B'(P(x))\circ P'(x)$

$P'(x) = (\Phi'(x)\xi, \Phi'(x)\eta)$ $\quad B'(\alpha,\beta)(u,v) = B(u,\beta) + B(\alpha,v)$

proof

If $B: U\times V \to W$ is bilinear, U, V, W vector spaces

then $B'(\alpha,\beta): U\times V \to W$ and

$$B'(\alpha,\beta)(u,v) = \lim_{t\to 0} \frac{B((\alpha,\beta)+t(u,v)) - B(\alpha,\beta)}{t} = \lim_{t\to 0}\frac{B(\alpha+tu, \beta+tv) - B(\alpha,\beta)}{t}$$

$$= \lim_{t \to 0} \frac{B(\alpha,\beta) + t\,B(u,\beta) + t\,B(\alpha,v) + t^2 B(u,v) - B(\alpha,\beta)}{t}$$

$$= B(u,\beta) + B(\alpha,v)$$

So $f'(1) = B'(P(1)) \circ P'(1) \quad : T_1Q \to T_1Q$

$$f'(1) = B'(\overset{=\mathrm{id}_{T_1Q}}{\Phi(1)}\xi, \Phi(1)\eta) \circ (\mathrm{ad}\,\xi, \mathrm{ad}\,\eta)$$

$$f'(1)(\zeta) = B'(\overset{\alpha}{\xi}, \overset{\beta}{\eta})(\overset{u}{\mathrm{ad}\,\xi(\zeta)}, \overset{v}{\mathrm{ad}\,\eta(\zeta)})$$

$$= B(\overset{u}{\mathrm{ad}\,\xi(\zeta)}, \overset{\beta}{\eta}) + B(\overset{\alpha}{\xi}, \overset{v}{\mathrm{ad}\,\eta(\zeta)})$$

$$= [\mathrm{ad}\,\xi(\zeta), \eta] + [\xi, \mathrm{ad}\,\eta(\zeta)]$$

$$= [[\xi,\zeta],\eta] + [\xi,[\eta,\zeta]]$$

Since $f = g$ $\quad f'(1) = g'(1)$ so

$f'(1)(\zeta) = g'(1)(\zeta)$ i.e.

$$\underbrace{[[\xi,\zeta],\eta]}_{\overset{?}{=} -[\eta,[\xi,\zeta]]} + [\xi,[\eta,\zeta]] = [[\xi,\eta],\zeta]$$

This is not the Leibniz identity : you need $[[\xi,\zeta],\eta] \overset{?}{=} -[\eta,[\xi,\zeta$