DERIVATIONS OF A FINITE DIMENSIONAL JB*-TRIPLE (AFTER MEYBERG)

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1. Lie algebras—Meyberg, Chapter 5

An algebra $\mathcal{L}$ with multiplication $(x,y) \mapsto [x,y]$ is a Lie algebra if

$$[xx] = 0$$

and

$$[[xy]z] + [[yz]x] + [[zx]y] = 0.$$

Left multiplication in a Lie algebra is denoted by $\mathrm{ad}(x)$: $\mathrm{ad}(x)(y) = [x,y]$. An associative algebra A becomes a Lie algebra A^- under the product, $[xy] = xy - yx$.

The first axiom implies that $[xy] = -[yx]$ and the second (called the *Jacobi identity*) implies that $x \mapsto \mathrm{ad}x$ is a homomorphism of $\mathcal{L}$ into the Lie algebra $(\mathrm{End}\,\mathcal{L})^-$, that is, $\mathrm{ad}\,[xy] = [\mathrm{ad}\,x, \mathrm{ad}\,y]$.

Assuming that $\mathcal{L}$ is finite dimensional, the Killing form is defined by $\lambda(x,y) = \mathrm{tr}\,(\mathrm{ad}(x)\mathrm{ad}(y))$.

Theorem 1 (CARTAN criterion—Theorem 1, page 41)**.** *A finite dimensional Lie algebra $\mathcal{L}$ over a field of characteristic 0 is semisimple if and only if the Killing form is nondegenerate.*

Proof. The proof is not given in Meyberg's notes. I might add a proof later (and the definition of semisimple) from Jacobson's Lie algebra book. However, we don't really need this theorem for our purposes since the Killing form will be nondegenerate in the case we are interested in (finite dimensional JB*-triples). □

A linear map D is a derivation if $D \cdot \mathrm{ad}(x) = \mathrm{ad}(Dx) + \mathrm{ad}(x) \cdot D$. Each $\mathrm{ad}(x)$ is a derivation, called an inner derivation. Let $\Theta(\mathcal{L})$ be the set of all derivations on $\mathcal{L}$.

Theorem 2 (Zassenhaus—Theorem 3, page 42)**.** *If the finite dimensional Lie algebra $\mathcal{L}$ over a field of characteristic 0 is semisimple (that is, its Killing form is nondegenerate), then every derivation is inner.*

Proof. Let D be a derivation of $\mathcal{L}$. Since $x \mapsto \mathrm{tr}\,(D \cdot \mathrm{ad}\,(x))$ is a linear form, there exists $d \in \mathcal{L}$ such that $\mathrm{tr}\,(D \cdot \mathrm{ad}\,(x)) = \lambda(d,x) = \mathrm{tr}\,(\mathrm{ad}\,(d) \cdot \mathrm{ad}\,(x))$. Let E be the derivation $E = D - \mathrm{ad}\,(d)$ so that

$$\mathrm{tr}\,(E \cdot \mathrm{ad}\,(x)) = 0. \tag{1}$$

Note next that

$$\begin{aligned} E\cdot[\mathrm{ad}\,(x),\mathrm{ad}\,(y)] &= E\cdot \mathrm{ad}\,(x)\cdot \mathrm{ad}\,(y) - E\cdot \mathrm{ad}\,(y)\cdot \mathrm{ad}\,(x)\\ &= (\mathrm{ad}\,(x)\cdot E + [E,\mathrm{ad}\,(x)])\cdot \mathrm{ad}\,(y) - E\cdot \mathrm{ad}\,(y)\cdot \mathrm{ad}\,(x)\end{aligned}$$

so that

$$\begin{aligned} [E,\mathrm{ad}\,(x)]\cdot \mathrm{ad}\,(y) &= E\cdot[\mathrm{ad}\,(x),\mathrm{ad}\,(y)] - \mathrm{ad}\,(x)\cdot E\cdot \mathrm{ad}\,(y) + E\cdot \mathrm{ad}\,(y)\cdot \mathrm{ad}\,(x)\\ &= E\cdot[\mathrm{ad}\,(x),\mathrm{ad}\,(y)] + [E\cdot \mathrm{ad}\,(y),\mathrm{ad}\,(x)]\end{aligned}$$

and

$$\mathrm{tr}\,([E,\mathrm{ad}\,(x)]\cdot \mathrm{ad}\,(y)) = \mathrm{tr}\,(E\cdot[\mathrm{ad}\,(x),\mathrm{ad}\,(y)]).$$

However, since E is a derivation

$$\begin{aligned} [E,\mathrm{ad}\,(x)]\cdot \mathrm{ad}\,(y) &= E\cdot \mathrm{ad}\,(x)\cdot \mathrm{ad}\,(y) - \mathrm{ad}\,(x)\cdot E\cdot \mathrm{ad}\,(y)\\ &= (\mathrm{ad}\,(Ex) + \mathrm{ad}\,(x)\cdot E)\cdot \mathrm{ad}\,(y) - \mathrm{ad}\,(x)\cdot E\cdot \mathrm{ad}\,(y)\\ &= \mathrm{ad}\,(Ex)\cdot \mathrm{ad}\,(y).\end{aligned}$$

Thus

$$\begin{aligned} \lambda(Ex,y) &= \mathrm{tr}\,(\mathrm{ad}\,(Ex)\cdot \mathrm{ad}\,(y))\\ &= \mathrm{tr}\,([E,\mathrm{ad}\,(x)]\cdot \mathrm{ad}\,(y))\\ &= \mathrm{tr}\,(E\cdot[\mathrm{ad}\,(x),\mathrm{ad}\,(y)]) = 0 \text{ by (1)}).\end{aligned}$$

Since x and y are arbitrary, $E=0$ and so $D-\mathrm{ad}\,(d)=0$. □

2. Derivations of Lie triple systems—Meyberg, Chapter 6

A Lie triple system is a vector space $\mathcal{F}$ together with a triple product $[\cdot,\cdot,\cdot]$ which satisfies

(1) $[xxz]=0$ (implies $[xyz]=-[yxz]$)
(2) $[xyz]+[yzx]+[zxy]=0$ (Jacobi identity)
(3) $[uv[xyz]]=[[uvx]yz]+[x[uvy]z]+[xy[uvz]]$

Examples:

- A Lie algebra $(\mathcal{L},[xy])$ under $[xyz]:=[[xy]z]$
- A subspace of a Lie algebra, closed under $[[xy]z]$
- An associative triple system $(\mathcal{F},<xyz>)$ under $[xyz]:=<xyz>-<yxz>-<zxy>+<zyx>$
- A Jordan algebra under $[xyz]:=[L(x),L(y)]z$

Define $L'(x,y),R'(z,y),P'(x,z)\in\mathrm{End}\,(\mathcal{F})$ by

$$[xyz]=L'(x,y)z=R'(z,y)x=P'(x,z)y,$$

where we are using L' (etc.) instead of L to avoid confusion with the operator $L(x,y)$ of a Jordan triple system ($L'(x,y)z=[xyz]$ in Lie triple systems, $L(x,y)z=\{xyz\}$ in Jordan triple systems).

The axioms for a Lie triple system become

(1) $L'(x,x)=0$ (implies $L'(x,y)=-L'(y,x)$)
(2) $L'(x,y)=R'(x,y)-R'(y,x)$

(3) $[L'(x,y), L'(u,v)] = L'([xyu], v) + L'(u, [xyv])$

A derivation is a linear map $D : \mathcal{F} \to \mathcal{F}$ satisfying $[D, L'(x,y)] = L'(Dx, y) + L'(x, Dy)$, equivalently, $D[xyz] = [Dx, y, z] + [x, Dy, z] + [x, y, Dz]$. From this it follows that if D, D' are derivations, then so is $[D, D']$, so that the set $\Theta(\mathcal{F})$ of all derivations on $\mathcal{F}$ is a Lie algebra of operators. Let $\mathcal{H}(\mathcal{F})$ = the span of all $L'(x,y)$.

The following lemma is immediate from the definition of derivation.

Lemma 2.1 (Lemma 2, page 44). *$L'(x,y)$ is a derivation of $\mathcal{F}$ and $\mathcal{H}(\mathcal{F})$ is an ideal in $\Theta(\mathcal{F})$*

Theorem 3 (Theorem 1, page 45). *Let $\mathcal{F}$ be a Lie triple system, let $\mathcal{G}$ be a subalgebra of $\Theta(\mathcal{F})$, and suppose that $\mathcal{H} \subset \mathcal{G}$.*

(i): *$\mathcal{L}(\mathcal{G}, \mathcal{F}) := \mathcal{G} \oplus \mathcal{F}$ is a Lie algebra under the product*

$$[H_1 \oplus x_1, H_2 \oplus x_2] = ([H_1, H_2] + L'(x_1, x_2)) \oplus (H_1 x_2 - H_2 x_1) \tag{2}$$

(ii): *$\theta'(H \oplus x) = (-H) \oplus x$ is an involution of $\mathcal{L}$, that is, $\theta'^2 = Id$ and $\theta'[X, Y] = [\theta' Y, \theta' X]$.*

(iii): *$\mathcal{L}(\mathcal{H}, \mathcal{F})$ is an ideal in $\mathcal{L}(\mathcal{G}, \mathcal{F})$*

(iv): *$[xyz] = [[x,y],z]$ for $x, y, z \in \mathcal{F}$.*

(v): *$\mathcal{F} = \{X \in \mathcal{L}(\mathcal{G}, \mathcal{F}) : \theta' X = X\}$*

Proof. If $X = H \oplus x$, then $[X, X] = ([H, H] + L'(x, x)) \oplus (Hx - Hx) = 0$. We have to show that $J(Y, Y, Z) := [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ for alll $X, Y, Z \in \mathcal{L}$. It is sufficient to show this for $X, Y, Z \in \mathcal{G} \cup \mathcal{F}$.

Since $\mathcal{G}$ is a subalgebra of $\Theta(\mathcal{F})$, $J(\mathcal{G}, \mathcal{G}, \mathcal{G}) = 0$.

If $H_i \in \mathcal{G}, x \in \mathcal{F}$ we get $[[H_1, H_2], x] = [H_1, H_2]x = H_1(H_2 x) - H_2(H_1(x) = [H_1, [H_2, x]] - [H_2, [H_1, x]]$.[1] This shows $J(\mathcal{G}, \mathcal{G}, \mathcal{F}) = 0$. Since $J(H_1, x, H_2) = J(H_2, H_1, x)$ and $J(x, H_1, H_2) = J(H_1, H_2, x)$, we have $J(\mathcal{G}, \mathcal{F}, \mathcal{G}) = 0$ and $J(\mathcal{F}, \mathcal{G}, \mathcal{G}) = 0$.

By (2), $[[H, x], y] + [[x, y], H] + [[y, H], x] = L(Hx, y) + [L(x, y), H] + L(x, Hy) = 0$ since H is a derivation. As above, $J(x, H, y) = J(H, y, x)$ and $J(x, y, H) = J(H, x, y)$ so that $J(\mathcal{F}, \mathcal{G}, \mathcal{F}) = J(\mathcal{F}, \mathcal{F}, \mathcal{G}) = 0$. Finally, by the Jacobi identity, $J(\mathcal{F}, \mathcal{F}, \mathcal{F}) = 0$.

This proves (i) and the other statements are trivially verified. □

3. Derivations on Jordan triple systems—Meyberg, Chapter 11

Let $(\mathcal{A}, P)$ be a Jordan triple system, that is, $P : \mathcal{A} \to \mathrm{End}(\mathcal{A})$ is a quadratic map, inducing the bilinear map $L : \mathcal{A} \times \mathcal{A} \to \mathrm{End}(\mathcal{A})$ via $L(x,y)z = \{xyz\} = P(x,z)y$ (note that $P(x) = P(x,x)/2$), and satisfying

JT1: $L(x,y)P(x) = P(x)L(y,x)$ "homotopy formula"

JT2: $L(P(x)y, y) = L(x, P(y)x)$

[1]Longer version: $[[H_1, H_2], x] = [[H_1 \oplus 0, H_2 \oplus 0], 0 \oplus x] = [[H_1, H_2] \oplus 0, 0 \oplus x] = 0 \oplus [H_1, H_2]x = 0 \oplus (H_1(H_2 x) - H_2(H_1 x)) = 0 \oplus H_1(H_2 x) - 0 \oplus H_2(H_1 x) = [H_1 \oplus 0, 0 \oplus H_2 x] - [H_2 \oplus 0, 0 \oplus H_1 x] = [H_1 \oplus 0, [H_2 \oplus 0, 0 \oplus x]] - [H_2 \oplus 0, [H_1 \oplus 0, 0 \oplus x]] = [H_1, [H_2, x]] - [H_2, [H_1, x]]$

JT3: $P(P(x)y) = P(x)P(y)P(x)$ "fundamental formula"

A derivation is a linear map $D : \mathcal{A} \to \mathcal{A}$ satisfying $[D, P(x)] = P(Dx, x)$. Let $\Theta(\mathcal{A})$ be the set of all derivations on $\mathcal{A}$. $\Theta(\mathcal{A})$ is a Lie subalgebra of $(\mathrm{End}\,(\mathcal{A}))^{-}$. By linearization, derivations satisfy

$$D\{xyz\} - \{x, Dy, z\} = \{Dx, y, z\} + \{x, y, Dz\}$$

or

$$[D, L(x, y)] = L(Dx, y) + L(x, Dy). \tag{3}$$

If the underlying field of $\mathcal{A}$ is of characteristic 0, then all equations in $\mathcal{A}$ are equivalent to what we are used to calling the "main identity," namely,

$$L(z, y)P(x) + P(x)L(y, z) = P(\{zyz\}, x). \tag{4}$$

For another proof of (4), see Meyberg pp. 107–108. From (4) we have the following lemma.

Lemma 3.1 (Lemma 1, page 108)**.** *$D(x, y) := L(x, y) - L(y, x)$ is a derivation.*

A derivation of $\mathcal{A}$ is inner if it is a finite sum of the $D(x, y)$.

Theorem 4 (Theorem 1, page 108)**.** *If $(\mathcal{A}, P)$ is a Jordan triple system, then $(\mathcal{A}, [xyz])$ is a Lie triple system, where $[xyz] = \{x, y, z\} - \{y, x, z\} = D(x, y)z$.*

Proof. It is obvious that $[xxz] = 0$ and the Jacobi identity follows from $\{uvw\} = \{wvu\}$. It remains to prove the operator form of the third axiom, namely $[L'(x, y), L'(u, v)] = L'([xyu], v) + L'(u, [xyv])$. To do this we will use Lemma 3.1 and (3). Actually $L' = D$ and so we have

$$\begin{aligned} [D(x, y), D(u, v)] &= [D(x, y), L(u, v) - L(v, u)] \\ &= L(D(x, y)u, v) + L(u, D(x, y)v) \\ &\quad - L(D(x, y)v, u) - L(v, D(x, y)u) \\ &= D(D(x, y)u, v) + D(u, D(x, y)v), \end{aligned}$$

as required. □

Let us recall that if $(\mathcal{A}, P)$ is a Jordan triple system, and $u \in \mathcal{A}$, then $(\mathcal{A}, P_u, {}^{(2,u)})$, where $P_u(x) = P(x)P(u)$ and $x^{(2,u)} = P(x)u$, is a quadratic Jordan algebra called the u-homotope of $\mathcal{A}$. (Theorem 1, page 94; proof on page 95). More generally, the following easily verified proposition leads to an important consequence (Corollary 3.3) of Theorem 4.

Proposition 3.2 (Theorem 2, page 96)**.** *If $(\mathcal{A}, P)$ is a Jordan triple system, and $V \in End(()\mathcal{A})$ satisfies $PV(x) = VP(x)V$, then $(\mathcal{A}, P_V)$, where $P_V(x) = P(x)V$, is a Jordan triple system called the V-homotope of $\mathcal{A}$.*

Corollary 3.3 (Corollary 3, page 109)**.** *$\mathcal{F} := \mathcal{A} \oplus \tilde{\mathcal{A}}$ is a Lie triple system under*

$$[(x_1, x_2), (y_1, y_2), (z_1, z_2)] = (\{x_1y_2z_1\} - \{y_1x_2z_1\}, \{x_2y_1z_2\} - \{y_2x_1z_2\}).$$

Proof. Applying Theorem 4 to any V-homotope $\mathcal{A}_V$ shows that $\mathcal{A}$, together with $[xyz] = \{x, Vy, z\} - \{y, Vx, z\}$ is a Lie triple system. In particular, if j is an involution (=automorphism of order 2) of $\mathcal{A}$, that is $j\{xyz\} = \{j(x), j(y), j(z)\}$ then $P(j(x)) = jP(x)j$ so that $\mathcal{A}$, together with $[xyz] = \{x, j(y), z\} - \{y, j(x), z\}$ is a Lie triple system.

The space $\mathcal{F}$ is a Jordan triple system under $\{(x_1, x_2), (y_1, y_2), (z_1, z_2)\} = (\{x_1y_1z_1\}, \{x_2y_2z_2\})$ and the exchange map $j(x_1, x_2) = (x_2, x_1)$ is an involution, proving the corollary. □

Note that for $x = x_1 \oplus \tilde{x}_2, y = y_1 \oplus \tilde{y}_2$ in the Lie triple system $\mathcal{A} \oplus \tilde{\mathcal{A}}$,

$$L'(x, y) = (L(x_1, y_2) - L(y_1, x_2), L(x_2, y_1) - L(y_2, x_1)) = \ell(x_1, y_2) + \ell(x_2, y_1).$$

Let $\mathcal{E} = \text{End}(\mathcal{A}) \oplus \text{End}\,(\mathcal{A})$ and consider the Lie algebra $\mathcal{E}^-$ with the product $[(A, B), (A', B')] = ([A, A'], [B, B'])$ and its subset

$$\gamma(\mathcal{A}) := \{(U, V) \in \mathcal{E} : UP(x) - P(x)V = P(Ux, x), VP(x) - P(x)U = P(Vx, x)\}$$

Note that $(I, -I)$, (D, D) and $\ell(x, y) := (L(x, y), -L(y, x))$ all belong to $\gamma(\mathcal{A})$, where I is the identity operator on $\mathcal{A}$ and $D \in \Theta(\mathcal{A})$. (Proof: $(I, -I) \in \gamma(\mathcal{A}) \Leftrightarrow 2P(x) = P(x, x)$; $(D, D) \in \gamma(\mathcal{A}) \Leftrightarrow DP(x) - P(x)D = P(Dx, x)$, which is the definition of derivation; $\ell(x, y) \in \gamma(\mathcal{A}) \Leftrightarrow L(x, y)P(x) + P(x)L(y, x) = P(L(x, y)x, x)$ and $L(y, x)P(x) + P(x)L(x, y) = P(L(y, x)x, x)$, both of which are immediate consequences of the main identity (4).)

We let $\mathcal{H}(\mathcal{A})$ be the submodule in $\gamma(\mathcal{A})$ spanned by all the $\ell(x, y)$. Note that $\mathcal{H}(\mathcal{A})$, defined here, is the same as $\mathcal{H}(\mathcal{A} \oplus \tilde{\mathcal{A}})$, defined for the Lie triple system $\mathcal{A} \oplus \tilde{\mathcal{A}}$; indeed $\mathcal{H}(\mathcal{A}) = \text{span}\,\{\ell(x, , y) : x, y \in \mathcal{A}\}$ and $\mathcal{H}(\mathcal{A} \oplus \tilde{\mathcal{A}}) = \text{span}\,\{\ell(x_1, y_2) + \ell(x_2, y_1) : x_1, x_2, y_1, y_2 \in \mathcal{A}\}$.

Lemma 3.4 (Lemma 2, page 110)**.** *$\gamma(\mathcal{A})$ is a subalgebra of $\mathcal{E}^-$ (called the structure algebra of $\mathcal{A}$) and $\mathcal{H}(\mathcal{A})$ is an ideal in $\gamma(\mathcal{A})$ (called the inner structure algebra of $\mathcal{A}$).*

Proof. If $(U, V) \in \gamma(\mathcal{A})$, then $UP(x + y) - P(x + y)V = P(Ux + Uy, x + y)$, which simplifies to $UP(x, y) - P(x, y)V = P(Uy, x) + P(Ux, y)$, and therefore, for $(U, V), (U', V') \in \gamma(\mathcal{A})$, $UP(U'x, x) - P(U'x, x)V = P(Ux, U'x) + P(UU'x, x)$, and

$$\begin{aligned} UU'P(x) &= U(P(x)V' + P(U'x, x)) = (UP(x))V' + UP(U'x, x) \\ &= (P(x)V + P(Ux, x))V' + P(UU'x, x) \\ &\quad + P(U'x, Ux) + P(U'x, x)V. \end{aligned}$$

Interchanging (U, V) and (U', V'), subtracting, and simplyfing yields

$$[U, U']P(x) = P(x)[V, V'] + P([U, U']x, x).$$

Similarly, one has

$$[V, V']P(x) = P(x)[U, U'] + P([V, V']x, x).$$

so $([U, U'], [V, V']) \in \gamma(\mathcal{A})$ proving that $\gamma(\mathcal{A})$ is a subalgebra of $\mathcal{E}^-$.

For $(U,V)\in\gamma(\mathcal{A})$ we know from the first part of the proof that

$$UP(x,y)z - P(x,y)Vz = P(Ux,y)z + P(x,Uy)z, \tag{5}$$

which is the same as

$$[U, L(x,z)] = L(Ux,z) + L(x,Vz), \tag{6}$$

and similarly

$$[V, L(x,z)] = L(Vx,z) + L(x,Uz). \tag{7}$$

With $\ell(u,v) = (L(u,v), -L(v,u)) \in \mathcal{H}(\mathcal{A})$, we have

$$\begin{aligned} [(U,V),\ell(u,v)] &= ([U,L(u,v), -[V,L(v,u])\\ &= (L(Uu,v)+L(u,Vv), -L(Vv,u)-L(v,Uu)\\ &= \ell(Uu,v)+\ell(u,Vv), \end{aligned}$$

so $\mathcal{H}(\mathcal{A})$ is an ideal. □

Lemma 3.5 (Lemma 3, page 111)**.** *$\gamma(\mathcal{A})$ is a subalgebra of the derivation algebra of the Lie triple system $\mathcal{A}\oplus\tilde{\mathcal{A}}$, that is, if $(U,V)\in\gamma(\mathcal{A})$, then $(U,V)(x_1\oplus\tilde{x}_2) := Ux_1\oplus V\tilde{x}_2$ is a derivation of the Lie triple system $\mathcal{A}\oplus\tilde{\mathcal{A}}$ defined in Corollary 3.3.*

Proof. By the previous lemma, $\gamma(\mathcal{A})$ is closed under brackets. We just need to show that $\gamma(\mathcal{A})\subset\Theta(\mathcal{A}\oplus\tilde{\mathcal{A}})$, which is to say that for $(U,V)\in\gamma(\mathcal{A})$ and $x=(x_1,x_2),\ y=(y_1,y_2)\in\mathcal{A}\oplus\tilde{\mathcal{A}}$, we have

$$[(U,V), L(x,y)] = L((U,V)x,y) + L(x,(U,V)y). \tag{8}$$

Applying the right hand side of (8) to $z=(z_1,z_2)$ yields

$$(\{Ux_1,y_2,z_1\} - \{y_1,Vx_2,z_1\}, \{Vx_2,y_1,z_2\} - \{y_2,Ux_1,z_2\}) \tag{9}$$

$$+(\{x_1,Vy_2,z_1\} - \{Uy_1,x_2,z_1\}, \{x_2,Uy_1,z_2\} - \{Vy_2,x_1,z_2\}). \tag{10}$$

Applying the left hand side of (8) to $z=(z_1,z_2)$ yields

$$(U\{x_1y_2z_1\} - U\{y_1x_2z_1\}, V\{x_2y_1z_2\} - V\{y_2,x_1,z_2\}) \tag{11}$$

$$-(\{x_1,y_2,Uz_1\} - \{y_1,x_2,Uz_1\}, \{x_2,y_1,Vz_2\} - \{y_2,x_1,Vz_2\}). \tag{12}$$

Let us rewrite these equations in terms of the quadratic operator P. We have respectively

$$(P(Ux_1,z_1)y_2 - P(y_1,z_1)Vx_2, P(Vx_2,z_2)y_1 - P(y_2,z_2)Ux_1) \tag{13}$$

$$+(P(x_1,z_1)Vy_2 - P(Uy_1,z_1)x_2, P(x_2,z_2)Uy_1 - P(Vy_2,z_2)x_1). \tag{14}$$

$$(UP(x_1,z_1)y_2 - UP(y_1,z_1)x_2, VP(x_2,z_2)y_1 - VP(y_2,z_2)x_1) \tag{15}$$

$$-(P(x_1,Uz_1)y_2 - P(y_1,Uz_1)x_2, P(x_2,Vz_2)y_1 - P(y_2Vz_2)x_1). \tag{16}$$

and we need to show that (13)+ (14)= (15)+ (16).

By two applications of (5), the sum of the first components of (13) and (14) equals the sum of the first components of (15) and (16). By two applications of the companion equation to (5), namely,

$$VP(x,y) - P(x,y)U = P(Vx,y) + P(x,Vy),$$

the sum of the second components of (13) and (14) equals the sum of the second components of (15) and (16). □

Theorem 5 (Theorem 2, page 112)**.** *If $\mathcal{A}$ is a Jordan triple system and $\mathcal{G}$ is a Lie subalgebra of $\gamma(\mathcal{A})$ containing $\mathcal{H}(\mathcal{A}) = \mathcal{H}(\mathcal{A} \oplus \tilde{\mathcal{A}})$, then $\mathcal{L} := \mathcal{L}(\mathcal{G}, \mathcal{A} \oplus \tilde{\mathcal{A}}) := \mathcal{G} \oplus \mathcal{A} \oplus \tilde{\mathcal{A}}$ is a Lie algebra (called the Koecher-Tits algebra of $(\mathcal{G}, \mathcal{A})$) under the product*

- **(i):** *the given product in $\mathcal{G}$*
- **(ii):** $[x_1 \oplus \tilde{x}_2, y_1 \oplus \tilde{y}_2] = S(x,y) := \ell(x_1, y_2) - \ell(y_1, x_2)$
- **(iii):** $[(U,V), x_1 \oplus \tilde{x}_2] = Ux_1 \oplus \widetilde{Vx_2}$.

Proof. Since $\mathcal{A} \oplus \tilde{\mathcal{A}}$ is a Lie triple system (by Corollary 3.3), since $\mathcal{H}(\mathcal{A}) = \mathcal{H}(\mathcal{A} \oplus \tilde{\mathcal{A}})$, and since $\gamma(\mathcal{A}) \subset \Theta(\mathcal{A} \oplus \tilde{\mathcal{A}})$ (by Lemma 3.5), this follows immediately from Theorem 3. □

Note that we are now using $S(x,y)$ to denote $L'(x_1 \oplus \tilde{x_2}, y_1 \oplus \tilde{y_2})$. For brevity's sake, we shall also denote $\mathcal{L}(\mathcal{G}, \mathcal{A} \oplus \tilde{\mathcal{A}})$ by $\mathcal{L}(\mathcal{G}, \mathcal{A})$.

The space $\gamma(\mathcal{A})$ has a canonical involutary authorphism $\theta : \gamma(\mathcal{A}) \to \gamma(\mathcal{A})$, $\theta(U,V) = (V,U)$, which is the restriction of the corresponding automorphism of $\mathcal{E}$. The derivation algebra $\Theta(\mathcal{A})$ of $\mathcal{A}$ may be identified via $D \leftrightarrow (D,D)$ with the fixed point set of θ:

$$\Theta(\mathcal{A}) \leftrightarrow \{X \in \gamma(\mathcal{A}) : \theta X = X\}.$$

Remark 3.6. Some properties of the product in Theorem 5.

- **(i):** $[\mathcal{A}, \mathcal{A}] = 0$, $[\mathcal{A}, \tilde{\mathcal{A}}] = 0$
  $([0 + (a + 0), 0 + (a' + 0)] = \ell(a,0) - \ell(a',0) = 0$, etc.)
- **(ii):** $[(U,V), a] = Ua$, $[(U,V), \tilde{b}] = \tilde{Vb}$
  $([(U,V) + (0+0), 0 + (a+0)] = Ua \oplus V(0) = Ua$, etc.)
- **(iii):** $[a, \tilde{b}] = \ell(a,b)$
  $([a \oplus 0, 0 \oplus \tilde{b})] = \ell(a, \tilde{b}) - \ell(0,0))$
- **(iv):** $[[a, \tilde{b}], c] = \{a,b,c\}$
  $([[a,\tilde{b}], c] = [\ell(a,b), c] = [(L(a,b), -L(b,a)) + (0+0), 0 + (c+0)] = L(a,b)c = \{abc\})$
- **(v):** θ extends to $\mathcal{L}$: $\theta(G \oplus a \oplus \tilde{b}) := \theta G \oplus b \oplus \tilde{a}$
  (assuming that $\theta(\mathcal{G}) \subset \mathcal{G}$)
- **(vi):** $j(G \oplus a \oplus \tilde{b}) := G \oplus (-a) \oplus (-\tilde{b})$ has (-1)-eigenspace $\mathcal{A} \oplus \tilde{\mathcal{A}}$
- **(vii):** $(\mathrm{ad}E)^3 = \mathrm{ad}E$, if $E = (I, -I) \in \mathcal{G}$
  $(\mathrm{ad}E(G + a + \tilde{b}) = 0 + a + (-\tilde{b})$; $(\mathrm{ad}E)^2(G + a + \tilde{b}) = 0 + a + \tilde{b}$; $(\mathrm{ad}E)^3(G + a + \tilde{b}) = 0 + a + (-\tilde{b}))$

(viii): $\mathcal{L}(\mathcal{H},\mathcal{A})$ is an ideal in $\mathcal{L}(\mathcal{G},\mathcal{A})$
(succinctly $\mathcal{L}(\mathcal{H}(\mathcal{A}\oplus\tilde{\mathcal{A}}),\mathcal{A}\oplus\tilde{\mathcal{A}})$ is an ideal in $\mathcal{L}(\mathcal{G},\mathcal{A}\oplus\tilde{\mathcal{A}})$)

Theorem 6. *Let $\mathcal{A}$ be an anisotropic ($\{xxx\}=0\Rightarrow x=0$) Jordan triple system and assume that $E=(I,-I)\in\mathcal{H}(\mathcal{A})$. Then*

$$\Theta(\mathcal{L}(\mathcal{H}(\mathcal{A}),\mathcal{A}))=\Theta(\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})).$$

Moreover, this set is equal to the set of all derivations of $\mathcal{L}(\mathcal{H}(\mathcal{A}),\mathcal{A})$ into $\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$. Each such derivation extends to an inner derivation of $\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$. Every derivation of $\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$ is inner and the Lie algebra of all derivations of $\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$ is isomorphic to $\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$.

Proof. In this proof, $\mathcal{G}$ denotes either $\mathcal{H}(\mathcal{A})$ or $\gamma(\mathcal{A})$. Let $D:\mathcal{L}(\mathcal{G},\mathcal{A})\to\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$ be a derivation;

$$D([X,Y])=[D(X),Y]+[X,D(Y)] \text{ for } X,Y\in\mathcal{L}(\mathcal{G},\mathcal{A}).$$

This makes sense because $\mathcal{L}(\mathcal{G},\mathcal{A})\subset\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$.

Since $E\in\mathcal{G}\subset\mathcal{L}(\mathcal{G},\mathcal{A})$, $D(E)\in\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$, so we can write $D(E)=S\oplus p\oplus\tilde{q}$, with $S=(S_1,S_2)\in\gamma(\mathcal{A})$, $p,q\in\mathcal{A}$.

Since $\mathcal{L}(\mathcal{G},\mathcal{A})$ is an ideal in $\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$, each $Y\in\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$ gives rise to a derivation $\operatorname{ad}Y$ of $\mathcal{L}(\mathcal{G},\mathcal{A})$ into $\mathcal{L}(\mathcal{G},\mathcal{A})$. Thus $D':=D+\operatorname{ad}(p\oplus(-\tilde{q}))$ is a derivation of $\mathcal{L}(\mathcal{G},\mathcal{A})$ into $\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$, and by Theorem 5(iii),

$$D'(E)=D(E)+[(p\oplus(-\tilde{q}),E]=S+p+\tilde{q}-p-\tilde{q}=S.$$

For $a\in\mathcal{A}$, write $D'a=(D'a)_{\gamma(\mathcal{A})}+(D'a)_1+(D'a)_2\in\gamma(\mathcal{A})\oplus\mathcal{A}\oplus\tilde{\mathcal{A}}$. Using Remark 3.6, we have

$$\begin{aligned} D'a &= D'([E,a])=[D'(E),a]+[E,D'a]\\ &= [D'(E),a]+[E,(D'a)_{\gamma(\mathcal{A})}+(D'a)_1+(D'a)_2]\\ &= [(S_1,S_2),a]+(D'a)_1-(D'a)_2=S_1a+(D'a)_1-(D'a)_2. \end{aligned}$$

Thus $(D'a)_1+(D'a)_2=S_1a+(D'a)_1-(D'a)_2$ and comparing components in $\mathcal{A}$ and $\tilde{\mathcal{A}}$, $D'a=(D'a)_1\in\mathcal{A}$. In particular, $S_1a=0$.

By a similar argument starting with $b\in\tilde{\mathcal{A}}$, we have $D'\tilde{b}\in\tilde{\mathcal{A}}$ and $S_2b=0$, so that $S=(S_1,S_2)=0$.

Define $U,V\in\operatorname{End}(\mathcal{A})$ by $U=D'|\mathcal{A}$ and $\tilde{V}b=D'\tilde{b}$. Then we have

$$D'(a\oplus\tilde{b})=Ua\oplus\tilde{V}b=[(U,V),a\oplus\tilde{b}]=\operatorname{ad}(U,V)(a\oplus\tilde{b}). \tag{17}$$

If $T=(T_1,T_2)\in\mathcal{G}$, then

$$D'[T,a\oplus\tilde{b}]=[D'(T),a\oplus\tilde{b}]+[T,Ua\oplus\tilde{V}b]. \tag{18}$$

If we write $D'(T)=(G_1,G_2)\oplus x\oplus\tilde{y}$ then the previous equation becomes

$$UT_1a\oplus V\tilde{T_2}b=(\ell(x,b)-\ell(a,y))\oplus(G_1a+T_1Ua)\oplus(G_2b\tilde{+}T_2Vb).$$

Comparing components leads to

$$(G_1,G_2)=[(U,V),(T_1,T_2)] \tag{19}$$

and $\ell(x,b) - \ell(a,y) = 0$ for every $a, b \in \mathcal{A}$. Thus $L(x,x) = L(y,y) = 0$ and since $\mathcal{A}$ is anisotropic, $x = y = 0$ so that $D'(\mathcal{G}) \subset \mathcal{G}$ in the case that $\mathcal{G} = \gamma(\mathcal{A})$.

In the case that $\mathcal{G} = \mathcal{H}(\mathcal{A})$, we also have $D'(\mathcal{G}) \subset \mathcal{G}$ since

$$D'\ell(x,y) = D'[x,\tilde{y}] = [D'x,\tilde{y}] + [x, D'\tilde{y}] = [Ux,\tilde{y}] + [x,\tilde{Vy}] = \ell(Ux,y) + \ell(x,Vy). \tag{20}$$

We now know that D' leaves $\mathcal{G}$ and $\mathcal{A} \oplus \tilde{\mathcal{A}}$ invariant and that by (17), $D' = \mathrm{ad}\,(U,V)$ on $\mathcal{A} \oplus \tilde{\mathcal{A}}$ and by (19) and (20), $D' = \mathrm{ad}\,(U,V)$ on $\mathcal{G}$.

Finally we show that $(U,V) \in \gamma(\mathcal{A})$. We have

$$[(U,V),\ell(x,y)] = (UL(x,y) - L(x,y)U, L(y,x)V - VL(y,x))$$

and from (20)

$$D'(\ell(x,y)) = (L(Ux,y) + L(x,Vy), -L(Vy,x) - L(y,Ux)).$$

The equality of the right sides of these two equations is the same as the validity of the two equations (6) and (7), which is equivalent to $(U,V) \in \gamma(\mathcal{A})$.

We have shown that, with $\mathcal{G} = \mathcal{H}(\mathcal{A})$ or $\mathcal{G} = \gamma(\mathcal{A})$, any derivation D from $\mathcal{L}(\mathcal{G},\mathcal{A})$ into $\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$ extends to an inner derivation of $\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$ into itself, namely $D = \mathrm{ad}\,((U,V) \oplus (-p) \oplus \tilde{q})$. Conversely, since $\mathcal{L}(\mathcal{G},\mathcal{A})$ is an ideal in $\mathcal{L}(\gamma(\mathcal{A}),\mathcal{A})$, these maps are derivations of $\mathcal{L}(\mathcal{G},\mathcal{A})$ to itself. This proves all the statements in the theorem save the isomorphism.

For any Lie algebra, by the Jacobi identity, the map $X \mapsto \mathrm{ad}\,X$ is a homomorphism into the Lie algebra of derivations. For the Lie algebras $\mathcal{L}(\mathcal{G},\mathcal{A})$ this map was just shown to be onto. Now suppose $X \in \mathcal{L}(\mathcal{G},\mathcal{A})$ and $\mathrm{ad}\,X = 0$. Then with $X = (G_1,G_2) \oplus x \oplus \tilde{y}$,

$$0 = [X,E] = [(G_1,G_2),(I,-I)] + [x \oplus \tilde{y},(I,-I)] = -x \oplus \tilde{y},$$

so that $X = (G_1,G_2)$, and therefore $0 = [X, a \oplus \tilde{b}] = G_1 a \oplus \tilde{G_2 b}$ and $X = 0$. □

Let $\sigma(a,b) = \mathrm{tr}(L(a,b))$. Then (by the main identity) $\sigma(\{x,y,z\},u) = \sigma(z,\{y,x,u\})$, a property called "associative" for bilinear forms. Interchanging x and z we get $\sigma(L(x,y)z,u) = \sigma(z,L(y,x)u)$.

Assume (from now on) that σ is nondegenerate and positive definite (which is the case for finite dimensional JB*-triples), and let A^* be the adjoint of $A \in \mathrm{End}(\mathcal{A})$. Then, by associativity of σ, $(L(x,y))^* = L(y,x)$ with respect to the inner product σ.

Lemma 3.7. *If σ is nondegenerate, then $E = (I,-I) \in \mathcal{H}(\mathcal{A})$.*

Proof. Define $xy^* \in \mathrm{End}(\mathcal{A})$ to be the rank one operator $xy^*(z) = \sigma(z,y)x$. Since the rank one maps xy^* generate $\mathrm{End}\,(\mathcal{A})$, we have in particular $I = \sum_i u_i v_i^*$. Then by the associativity of σ,

$$\sigma(x,y) = \mathrm{tr}\,L(x,y)I = \sum \mathrm{tr}\,\{xyu_i\}v_i^* = \sum \sigma(\{xyu_i\},v_i) = \sigma(\sum\{u_iv_ix\},y).$$

Since σ is nondegenerate, $x = \sum L(u_i v_i)x$, so that $I = \sum L(u_i, v_i)$ and $(I, -I) = \sum(L(u_i, v_i), -L(u_i, v_i) \in \mathcal{H}(\mathcal{A})$. $\square$

Theorem 7 (Theorem 7, page 120)**.** *The Killing form λ of the Koecher-Tits algebra $\mathcal{L}(\mathcal{H}(\mathcal{A}), \mathcal{A})$ is nondegenerate if and only if σ is nondegenerate. If the characteristic of the underlying field is 0, then $\mathcal{L}(\mathcal{H}(\mathcal{A}), \mathcal{A})$ is semisimple if and only if $\mathcal{A}$ is semisimple if and only if σ is nondegenerate.*

Proof. We shall not give a proof of this theorem at this time. $\square$

Corollary 3.8. *If σ is nondegenerate, then $\mathcal{H}(\mathcal{A}) = \gamma(\mathcal{A})$.*

Proof. Because of Lemma 3.7, we can use Theorems 6 and 7 to show that $\gamma(\mathcal{A}) \subset \mathcal{H}(\mathcal{A})$. If $(U, V) \in \gamma(\mathcal{A})$, then $\operatorname{ad}(U, V)$ is a derivation of $\mathcal{L}(\gamma(\mathcal{A}), \mathcal{A})$ and hence an inner derivation of $\mathcal{L}(\mathcal{H}(\mathcal{A}), \mathcal{A})$, so we can write $\operatorname{ad}(U, V) = \operatorname{ad}((H_1, H_2) \oplus a \oplus \tilde{b})$ for some $(H_1, H_2) \in \mathcal{H}(\mathcal{A})$. Applying this equality to $x_1 \oplus \tilde{x_2}$, we have

$$Ux_1 \oplus \tilde{Vx_2} = (\ell(a, x_2) - \ell(x_1, b) \oplus H_1 x_1 \oplus \tilde{H_2 x_2}$$

proving that $(U, V) = (H_1, H_2) \in \mathcal{H}(\mathcal{A})$. $\square$

Corollary 3.9. *If σ is nondegenerate, then every derivation of $\mathcal{A}$ is inner.*

Proof. If $D \in \Theta(\mathcal{A})$, then $(D, D) \in \gamma(\mathcal{A}) = \mathcal{H}(\mathcal{A})$ so that $(D, D) = \sum_i \ell(u_i, v_i)$. Hence $D = \sum_i L(u_i, v_i) = -\sum_i L(v_i, u_i)$ and $D = \frac{1}{2}(D + D) = \sum_i (L(u_i, v_i) - L(v_i, u_i))$. $\square$

Theorem 8. *Every derivation of a finite dimensional JB*-triple is inner.*

Proof. We just need to verify that σ is nondegenerate. $\square$

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