Elementary Analysis Math 140B—Winter 2007
Solutions to First Midterm; February 7, 2007

Problem 1 (25 points) *Let* $f_n(x) = x^2/(nx+1)$ *if* $0 < x < \infty$, $n = 1, 2, \ldots$.

(a) *Does* f_n *converge uniformly on* $(0, 1)$*? Justify your answer.*

Solution: *YES;*

$$f_n(x) = \frac{x^2}{nx+1} = \frac{x^2/n}{x+1/n} \le \frac{x^2/n}{x} = x/n.$$

So if $0 < x < 1$, *we have* $\sup_{x\in(0,1)} |f_n(x)| \le 1/n \to 0$.

(b) *Does* f_n *converge uniformly on* $(1, \infty)$*? Justify your answer.*

Solution: *NO; Suppose the contrary. Then for any* $\epsilon > 0$, *there exists* $N \ge 1$ *such that*

$$\frac{x^2}{nx+1} < \epsilon \text{ for all } n > N \text{ and all } x > 1.$$

Now fix an $n > N$ *and let* $x \to \infty$ *to get*

$$\frac{x^2}{nx+1} = \frac{1}{n/x + 1/x^2} \to \infty,$$

a contradiction.

Problem 2 (25 points) *Let* $f_n(x) = |\sin \pi x|^n$ *if* $0 < x < \infty$, $n = 1, 2, \ldots$.

(a) *Does* f_n *converge uniformly on* $(0, 1)$*? Justify your answer.*

Solution: *NO; The pointwise limit is* $f(x) = 0$ *for* $x > 0, x \ne 1/2, 3/2, 5/2, \ldots$ *and* $f(1/2) = f(3/2) = \ldots = 1$ *which is not continuous on* $(0, 1)$.

(b) *Does* f_n *converge uniformly on* $(3/4, 5/4)$*? Justify your answer.*

Solution: *YES;*

$$\sup_{x\in(3/4,5/4)} |\sin \pi x|^n \le [\sin(3\pi/4)]^n \to 0.$$

Problem 3 (25 points) *Consider the power series* $\sum_{n=1}^{\infty} \frac{n}{n+1} x^n$.

(a) *What is the interval of convergence of this series?*

Solution: *The radius of convergence is 1, since* $[n/(n+1)]^{1/n} \to 1$ *(root test), or*

$$\frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} \to 1 \text{ (ratio test).}$$

The series diverges at $x = 1$ *and at* $x = -1$, *since* $n/(n+1) \not\to 0$ *and* $(-1)^n n/(n+1) \not\to 0$. *Thus the interval of convergence is* $(-1, 1)$.

(b) *Show that the series does not converge uniformly on its interval of convergence.*

Hint: Use the fact that a series $\sum_n g_n(x)$ *converges uniformly on a set* S*, then*

$$\lim_{n\to\infty}[\sup_{x\in S}|g_n(x)|] = 0.$$

Solution:

$$\lim_{n\to\infty}\left[\sup_{-1<x<1}\left|\frac{n}{n+1}x^n\right|\right] = \lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0.$$

Alternate solution: supposing to the contrary that

$$\frac{n}{n+1}x^n < \epsilon \text{ for all } n > N \text{ and all } |x| < 1,$$

fix $n > N$ *and let* $x \to 1$ *to get the contradiction* $n/(n+1) \leq \epsilon$.

Problem 4 (25 points) *Assume that* f_n *converges uniformly to* f *on* S *and that each* f_n *is bounded on* S*, that is*

$$M_n = \sup_{x\in S}|f_n(x)| < \infty \quad \text{for } n = 1, 2, \ldots$$

(a) *Show that* f *is bounded.*

Solution: *For any* $\epsilon > 0$*, choose* N *so that* $|f_n(x) - f(x)| < \epsilon$ *for* $n > N$ *and* $x \in S$*. Then for a fixed* $n > N$ *and all* $x \in S$*,*

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < \epsilon + M_n.$$

Thus $\sup_{x\in S}|f(x)| \leq \epsilon + M_n$ *so* f *is bounded.*

(b) *Show that there is a constant* $M > 0$ *such that* $M_n \leq M$ *for all* $n \geq 1$*.*

Solution: *By part (a), there is an* $M' > 0$ *such that* $|f(x)| \leq M'$ *for all* $x \in S$*. For any* $\epsilon > 0$*, choose* N *so that* $|f_n(x) - f(x)| < \epsilon$ *for* $n > N$ *and* $x \in S$*. Then for all* $n > N$ *and all* $x \in S$*,*

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq \epsilon + M'$$

so that $M_n \leq \epsilon + M'$ *for all* $n > N$*. Let* $M = \max\{M_1, M_2, \ldots, M_N, \epsilon + M'\}$*. Then* $M_n \leq M$ *for all* $n \geq 1$*.*

(c) *Assume that* f_n *and* g_n *are sequences of bounded functions on a set* S *and that both sequences converge uniformly on* S*. Show that* $f_n g_n$ *converges uniformly on* S*.*

Solution: *By part (a), there is a constant* K_1 *such that* $|g(x)| \leq K_1$ *for all* $x \in S$*. By part (b), there is a constant* K_2 *such that* $|f_n(x)| \leq K_2$ *for all* $n \geq 1$ *and all* $x \in S$*. Choose* N_1 *such that* $|g_n(x) - g(x)| < \epsilon/2(K_1 + K_2)$ *for all* $n > N_1$ *and all* $x \in S$ *and choose* N_2 *such that* $|f_n(x) - f(x)| < \epsilon/2(K_1 + K_2)$ *for all* $n > N_2$ *and all* $x \in S$*. Then for all* $n > \max\{N_1, N_2\}$ *and all* $x \in S$*,*

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| \\
&= |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\
&< K_2\frac{\epsilon}{2(K_1 + K_2)} + K_1\frac{\epsilon}{2(K_1 + K_2)} < \epsilon.
\end{aligned}$$