**Elementary Analysis Math 140B—Winter 2007**
**Solutions to Second Midterm; March 7, 2007**

**Problem 1 (25 points)** *Let* $f(x) = x^4 \sin \frac{1}{x}$ *for* $x \neq 0$ *and* $f(0) = 0$.

**(a)** *Show that* $f''(x)$ *exists for all values of* $x$.

**Solution:** *For* $x \neq 0$,

$$f'(x) = -x^2 \cos \frac{1}{x} + 4x^3 \sin \frac{1}{x}$$

*and for* $x = 0$,

$$f'(0) = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} h^3 \sin \frac{1}{h} = 0,$$

*so* $f'(x)$ *exists for all* $x$.

*For* $x \neq 0$,

$$f''(x) = \sin \frac{1}{x} - 6x \cos \frac{1}{x} + 12x^2 \sin \frac{1}{x}$$

*and*

$$f''(0) = \lim_{h\to 0, h\neq 0} \frac{f'(h) - f'(0)}{h} = \lim_{h\to 0} \left( -h \cos \frac{1}{h} + 4h^2 \sin \frac{1}{h} \right) = 0,$$

*so* $f''(x)$ *exists for all* $x$.

**(b)** *Is* $f''$ *continuous at* $x = 0$*? Justify your answer.*

**Solution:** *NO;* $\lim_{h\to 0} f''(h)$ *doesn't exist.*

**Problem 2 (25 points)** *Define the function* $f$ *by* $f(2^{-n}) = 2^{-(n+1)}$ *for* $n = 1, 2, \ldots$ *and* $f(x) = 0$ *for other values of* $x$.

**(a)** *Show that* $f$ *is continuous at* $x = 0$.

**Solution:** *By definition of* $f$, $|f(x)| \leq x/2$ *for every* $x$. *Hence* $\lim_{x\to 0} |f(x)| \leq \lim_{x\to 0} |x|/2 = 0$, *so* $\lim_{x\to 0} f(x) = 0$. *Since* $f(0) = 0$, $f$ *is continuous at 0.*

**(b)** *Is* $f$ *differentiable at* $x = 0$? *Justify your answer.*

**Solution:** *NO; if* $f'(0) = \lim_{h\to 0} \frac{f(h)}{h}$ *existed, then for* $h_n = 2^{-n}$, *we would have* $f'(0) = \lim_{n\to\infty} \frac{f(h_n)}{h_n} = \lim_{n\to\infty} (\frac{1}{2}) = \frac{1}{2}$ *and for* $h_n = 3^{-n}$ *we would have* $f'(0) = \lim_{n\to\infty} \frac{f(h_n)}{h_n} = \lim_{n\to\infty} (0) = 0$.

**Problem 3 (25 points)** *Let* $f$ *and* $g$ *be differentiable on* $(a, b)$ *and continuous on* $[a, b]$ *and suppose that* $g'(x) > 0$ *for all* $x \in (a, b)$.

**(a)** *Show that* $g(c) < g(b)$ *for every* $c \in (a, b)$.

Hint: Consider two points $x_1, x_2$ such tht $c < x_1 < x_2 < b$.

**Solution: (Using the hint)** *First,* $g(x_1) < g(x_2)$. *Now, since* $g$ *is continuous at* $b$,

$$g(x_1) \leq \lim_{x_2\to b} g(x_2) = g(b).$$

*Thus for every* $c, x_2 \in (a, b)$ *with* $c < x_2$,

$$g(c) < g(x_2) \leq g(b).$$

**Another solution (not using the hint)** *By the Mean Value Theorem,*

$$\frac{g(b) - g(c)}{b - c} = g'(d) \text{ for some } d \in (c, b).$$

*Since* $g'(d) > 0$ *and* $b - c > 0$ *we have* $g(b) - g(c) > 0$.

**(b)** *Show that there is $c \in (a,b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(c)-f(a)}{g(b)-g(c)}.$$

Hint: Consider the function $[f(x)-f(a)][g(b)-g(x)]$.

**Solution:** *Let $h(x) = [f(x)-f(a)][g(b)-g(x)]$. Then $h(a) = 0$ and $h(b) = 0$ so by Rolle's theorem, there exists $c \in (a,b)$ with $h'(c) = 0$. But*

$$h'(c) = -[f(c)-f(a)]g'(c) + f'(c)[g(b)-g(c)].$$

**Problem 4 (25 points)** *Suppose that $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$.*

**(a)** *Suppose that for each $x \in (a,b)$, there is an open interval containing $x$ on which $f$ is increasing. Prove that $f$ is increasing on $(a,b)$.*

**Solution:** *Let $x \in (a,b)$. By the assumption, there is $\delta > 0$ such that $f$ is increasing on $(x-\delta, x+\delta)$. Thus if $0 < h < \delta$, we have $f(x+h) - f(x) \geq 0$ so that*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

*is non-negative. Since we have shown that $f'(x) \geq 0$ for every $x \in (a,b)$, $f$ is increasing on $(a,b)$.*

**Another solution (using compactness, but a bit tedious to describe[1])** *Fix $a < c < d < b$. You want to show that $f(c) \leq f(d)$. For each $x \in [c,d]$, there is an open interval containing $x$ on which $f$ is increasing. The collection of all such open intervals forms an open cover of the compact set $[c,d]$, so it follows that $[c,d]$ is contained in the union of finitely many of these intervals, say $[c,d] \subset I_1 \cup \cdots \cup I_m$. Let $I_1$ denote one of these intervals which contains c, and let $x_1 =$ the right endpoint of $I_1$ so that $f(c) \leq f(x_1)$.* (More precisely, $f(c) \leq \lim_{x \to x_1 -} f(x)$, but let's be reasonable!) *If $d \in I_1$, then since $f$ is increasing on $I_1$, $f(c) \leq f(d)$ and we are done. Otherwise, since $x_1 \notin I_1$, let us denote by $I_2$ an interval which contains $x_1$, and let $x_2$ be the right endpoint of $I_2$. If $d \in I_2$, then since $f$ is increasing on $I_2$, $f(c) \leq f(x_1) \leq f(d)$ and we are done. Otherwise, since $x_2 \notin I_2$, let us denote by $I_3$ an interval which contains $x_2$, and let $x_3$ be the right endpoint of $I_3$. Continuing in this way we come (after a finite number of steps) to an interval containing $d$. Thus we have a sequence $c < x_1 < \cdots < x_k < d$ with $f(c) \leq f(x_1) \leq \cdots \leq f(x_k) \leq f(d)$.*

**(b)** *Suppose that $f'(x) \geq 0$ for $a < x < b$ and that $f'(x) > 0$ for at least one point in $(a,b)$. Prove that $f(a) < f(b)$.*

**Solution:** *We first note that $f$ is increasing on $(a,b)$ since $f'(x) \geq 0$ for all $x \in (a,b)$. We next establish that $f(a) \leq f(d)$ for any $d \in (a,b)$. Indeed, $f(a+\epsilon) \leq f(d)$ for $\epsilon$ small enough, and so*

$$f(a) = \lim_{\epsilon \to 0+} f(a+\epsilon) \leq f(d).$$

*Similary $f(d) \leq f(b)$ for all $d \in (a,b)$.*

*Now let $c$ be the point in $(a,b)$ with $f'(c) > 0$. It follows that for $h$ sufficiently small and positive, $f(c-h) < f(c+h)$ (that is, $f$ is strictly increasing at $c$). Thus*

$$f(a) \leq f(c-h) < f(c+h) \leq f(b).$$

[1] Don't read this if you have a weak heart!