EVOLUTION ALGEBRA

Freshman Seminar
University of California, Irvine

Bernard Russo

University of California, Irvine

Winter 2015

Table of Contents

Linear Algebras

(Ref.—Dickson, "Linear Algebras", 1914, pp.1–6)

Two couples of real numbers (a, b) and (c, d) are called equal if $a = c, b = d$.

Addition, subtraction and multiplication of two couples are defined by

- $(a, b) + (c, d) = (a + c, b + d)$
- $(a, b) - (c, d) = (a - c, b - d)$
- $(a, b)(c, d) = (ac - bd, ad + bc)$

Addition is seen to be commutative and associative:

- $x + x' = x' + x$, $(x + x') + x'' = x + (x' + x'')$

where x, x', x'' are any couples, $x = (a, b)$, $x' = (a', b')$, $x'' = (a'', b'')$.

Multiplication is commutative, associative, and distributive:

- $xx' = x'x$, $(xx')x'' = x(x'x'')$ (xx is denoted by x^2)
- $x(x' + x'') = xx' + xx''$, $(x' + x'')x = x'x + x''x$

Division is defined as the operation inverse to multiplication. Division except by (0,0) is possible and unique:

$$\frac{(c,d)}{(a,b)} = \left(\frac{ac+bd}{a^2+b^2}, \frac{ad-bc}{a^2+b^2}\right)$$

In particular we have

- $(a,0) \pm (c,0) = (a \pm c, 0)$, $(a,0)(c,0) = (ac,0)$, $\frac{(c,0)}{(a,0)} = \left(\frac{c}{a}, 0\right)$

Hence the couples $(a,0)$ combine under the above defined addition, multiplication, etc. exactly as the real numbers a combine under ordinary addition, multiplication, etc.

Thus, there is no danger in identifying the couple $(a,0)$ with the real number a, just as we identify the natural numbers among the signed integers, the integers among the rational numbers, and the latter among the real numbers

If, for brevity, you write $i = (0,1)$, then $i^2 = (0,1)(0,1) = (-1,0) = -1$ and you get the **complex numbers**: $(a,b) = (a,0) + (0,b) = a + (b,0)(0,1) = a + bi$

A set of complex numbers is called a **number field** if the sum, difference, product, and quotient (the divisor not being zero) of any two equal or distinct numbers of the set must be numbers belonging to the set.

Examples: complex numbers, real numbers, rational numbers.
(The set of integers is not a number field)

The concept of **matrix** affords an excellent example of a **linear algebra**. We can consider square matrices of n rows and n columns. For convenience, we take $n = 2$. Let

$$m = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \mu = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

be two matrices, where the elements $a, b, c, d, \alpha, \beta, \gamma, \delta$ belong to a fixed number field F, which will usually be the real numbers.

We say that m and μ are equal if their corresponding elements are equal, $a = \alpha$, etc. Addition and multiplication are defined by

$$m + \mu = \begin{bmatrix} a + \alpha & b + \beta \\ c + \gamma & d + \delta \end{bmatrix}, \ m\mu = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}$$

Addition is commutative and associative

$$x + x' = x' + x\ ,\ (x + x') + x'' = x + (x' + x'')$$

where x, x', x'' are any matrices of the same size.

Multiplication is associative and distributive

- $(xx')x'' = x(x'x'')$
- $x(x' + x'') = xx' + xx''\ ,\ (x' + x'')x = x'x + x''x$

However, multiplication of matrices is not commutative, and division m/μ is not always possible, even if

$$\mu \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Consider the four special matrices

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\ ,\ e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\ ,\ e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\ ,\ e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Their sixteen possible products by twos can be summarized as

$$e_{ij}e_{tk} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ if } t \neq j \qquad \text{and} \qquad e_{ij}e_{jk} = e_{ik} \tag{1}$$

Table 1 $X \times Y$

		Y			
	X $\times$ **Y**	e_{11}	e_{12}	e_{21}	e_{22}
X	e_{11}	e_{11}	e_{12}	0	0
	e_{12}	0	0	e_{11}	e_{12}
	e_{21}	e_{21}	e_{22}	0	0
	e_{22}	0	0	e_{21}	e_{22}

Table 2 $Y \times X$

		Y			
	Y $\times$ **X**	e_{11}	e_{12}	e_{21}	e_{22}
X	e_{11}	e_{11}	0	e_{21}	0
	e_{12}	e_{12}	0	e_{22}	0
	e_{21}	0	e_{11}	0	e_{21}
	e_{22}	0	e_{12}	0	e_{22}

If $m = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix and e is a number, we define the product em to be

$$em = e \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea & eb \\ ec & ed \end{bmatrix}$$

We now have

- $m = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ae_{11} + be_{12} + ce_{21} + de_{22}$
- $\mu = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \alpha e_{11} + \beta e_{12} + \gamma e_{21} + \delta e_{22}$
- $m + \mu = (a + \alpha)e_{11} + (b + \beta)e_{12} + (c + \gamma)e_{21} + (d + \delta)e_{22}$
- $m\mu = (a\alpha + b\gamma)e_{11} + (a\beta + b\delta)e_{12} + (c\alpha + d\gamma)e_{21} + (c\beta + d\delta)e_{22}$

The set of **hyper-complex numbers** $ae_{11} + be_{12} + ce_{21} + de_{22}$, in which a, b, c, d range independently over a field F, and for which addition and multiplication are defined as above is an example of a **linear associative algebra** over F with the four units $e_{11}, e_{12}, e_{21}, e_{22}$ subject to the multiplication table (1)

Consider the set of n-tuples $(x_1, \ldots, x_n)$, whose coordinates $x_1 \ldots, x_n$ range independently over a given number field F.

Two n-tupes are called equal if their corresponding coordinates are equal. Addition and subtraction of n-tuples are defined by

$$(x_1, \ldots, x_n) \pm (y_1, \ldots, y_n) = (x_1 \pm y_1, \ldots, x_n \pm y_n) \tag{2}$$

The product of any number ρ of the field F and any n-tuple $x = (x_1, \ldots, x_n)$ is defined to be

$$\rho x = x\rho = (\rho x_1, \ldots, \rho x_n) \tag{3}$$

The n units are defined to be

$$e_1 = (1, 0, \ldots, 0)\ , e_2 = (0, 1, \ldots, 0),\ \ldots\ , e_n = (0, \ldots, 0, 1)$$

Hence any n-tuple can be expressed in the form

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

A linear algebra is obtained by assuming that any two n-tuples

$$x = x_1e_1 + x_2e_2 + \cdots + x_ne_n \text{ and } y = y_1e_1 + y_2e_2 + \cdots + y_ne_n$$

can be combined by an operation called multiplication, which is subject to the distributive laws

$$x(y+z) = xy + xz \ , \ (y+z)x = yx + zx$$

Thus

$$xy = x_1y_1e_1e_1 + x_1y_2e_1e_2 + \cdots + x_iy_je_ie_j + \cdots + x_ny_ne_ne_n$$

The product xy is determined once we know the particular products among the units, that is, for fixed i and j, the coordinates $\gamma_{ij1}, \gamma_{ij2}, \ldots, \gamma_{ijn}$ of e_ie_j;

$$e_ie_j = \gamma_{ij1}e_1 + \gamma_{ij2}e_2 + \cdots + \gamma_{ijn}e_n$$

Properties (2) and (3) of n-tuples give

$$x \pm y = (x_1 \pm y_1)e_1 + \cdots + (x_n \pm y_n)e_n \text{ and } \rho x = x\rho = (\rho x_1)e_1 + \cdots + (\rho x_n)e_n$$

Every linear algebra of dimension n is nothing more than a set of n^3 numbers γ_{ijk}, where i, j, k range independently over the integers $1, 2, \ldots, n$.

$n = 2, n^3 = 8$

- $e_1^2 = e_1 e_1 = \gamma_{111} e_1 + \gamma_{112} e_2$
- $e_1 e_2 = \gamma_{121} e_1 + \gamma_{122} e_2$
- $e_2 e_1 = \gamma_{211} e_1 + \gamma_{212} e_2$
- $e_2^2 = e_2 e_2 = \gamma_{221} e_1 + \gamma_{222} e_2$

$$\text{If } x = x_1 e_1 + x_2 e_2 \text{ and } y = y_1 e_1 + y_2 e_2$$

then the product xy has coordinates z_1, z_2 given by

$$z_1 = x_1 y_1 \gamma_{111} + x_1 y_2 \gamma_{121} + x_2 y_1 \gamma_{211} + x_2 y_2 \gamma_{221}$$

$$z_2 = x_1 y_1 \gamma_{112} + x_1 y_2 \gamma_{122} + x_2 y_1 \gamma_{212} + x_2 y_2 \gamma_{222}$$

That is,

$$xy = z_1 e_1 + z_2 e_2$$

Genetic motivation

(Ref.—Reed, "Algebraic Structure of Genetic Inheritance", 1997, pp. 107-108)

Before we discuss the mathematics of genetics, we need to acquaint ourselves with the necessary language from biology.

A vague, but nevertheless informative, definition of a **gene** is simply a unit of hereditary information. The genetic code of an organism is carried on **chromosomes**.

Each gene on a chromosome has different forms that it can take. These forms are called **alleles**. E.g., the gene which determines blood type in humans has three different alleles, A, B, and O.

Since humans are **diploid** organisms (meaning we carry a double set of chromosomes—one from each parent), blood types are determined by two alleles.

Haploid cells (or organisms) carry a single set of chromosomes.

When diploid organisms reproduce, a process called **meiosis** produces **gametes** (sex cells) which carry a single set of chromosomes.

When these gamete cells fuse (e.g., when sperm fertilizes egg), the result is a **zygote**, which is again a diploid cell, meaning it carries its hereditary information in a double set of chromosomes.

When gametes fuse (or reproduce) to form zygotes a natural "multiplication" operation occurs.

As a natural first example, we consider simple Mendelian inheritance for a single gene with two alleles A and a.

In this case, two gametes fusing (or reproducing) to form a zygote gives the multiplication table shown in the following Table, which in freshman biology class might be called a Punnett square.

Table 3. Alleles passing from gametes to zygotes

	A	a
A	AA	Aa
a	aA	aa

The zygotes AA and aa are called **homozygous**, since they carry two copies of the same allele.

In this case, simple Mendelian inheritance means that there is no chance involved as to what genetic information will be inherited in the next generation; i.e., AA will pass on the allele A and aa will pass on a.

However, the zygotes Aa and aA (which are equivalent) each carry two different alleles. These zygotes are called **heterozygous**.

The rules of simple Mendelian inheritance indicate that the next generation will inherit either A or a with equal frequency. So, when two gametes reproduce, a multiplication is induced which indicates how the hereditary information will be passed down to the next generation.

This multiplication is given by the following rules:

1. $A \times A = A$
2. $A \times a = \frac{1}{2}A + \frac{1}{2}a$
3. $a \times A = \frac{1}{2}a + \frac{1}{2}A$
4. $a \times a = a$

Rules (1) and (4) are expressions of the fact that if both gametes carry the same allele, then the offspring will inherit it.

Rules (2) and (3) indicate that when gametes carrying A and a reproduce, half of the time the offspring will inherit A and the other half of the time it will inherit a.

These rules are an algebraic representation of the rules of simple Mendelian inheritance. This multiplication table is shown in Table 4.

Table 4. Multiplication table of the gametic algebra for simple Mendelian inheritance

	A	a
A	A	$\frac{1}{2}$(A+a)
a	$\frac{1}{2}$(a+A)	a

We should point out that we are only concerning ourselves with **genotypes** (gene composition) and not **phenotypes** (gene expression). Hence we have made no mention of the dominant or recessive properties of our alleles.

Now that we've defined a multiplication on the symbols A and a we can mathematically dedine the two dimensional algebra over R with basis {A, a} and multiplication table as in Table 4. This algebra is called the **gametic algebra** for simple Mendelian inheritance with two alleles.

But gametic multiplication is just the beginning! In order for actual diploid cells (or organisms) to reproduce, they must first go through a reduction division process so that only one set of alleles is passed on.

For humans this occurs when males produce sperm and females produce eggs. When reproduction occurs, the hereditary information is then passed on via the gametic multiplication we've already defined.

Therefore, when two zygotes reproduce, another multiplication operation is formed taking into consideration both the reduction division process and gametic multiplication.

In our example of simple Mendelian inheritance for one gene with the two alleles A and a, zygotes have three possible genotypes: AA, aa,and Aa.

Let's consider the case of two zygotes both with genotype Aa reproducing. The reduction division process splits the zygote and passes on one allele for reproduction.

In the case of simple Mendelian inheritance the assumption is that both alleles will be passed on with equal frequency. Thus, half the time A gets passed on and half the time a does.

We represent this with the "frequency distribution" $\frac{1}{2}A + \frac{1}{2}a$. Therefore, symbolically $Aa \times Aa$ becomes

$$\left(\frac{1}{2}A + \frac{1}{2}a\right) \times \left(\frac{1}{2}A + \frac{1}{2}a\right)$$

Formally multiplying these two expressions together results in

$$\frac{1}{4}AA + \frac{1}{2}Aa + \frac{1}{4}aa$$

using the notion that aA = Aa.

In this way, zygotic reproduction produces the multiplication table shown in Table 5. So we can define the three dimensional algebra over R with basis {AA, Aa, aa} and multiplication table as in Table 5. It is called the **zygotic algebra** for simple Mendelian inheritance with two alleles.

Table 5. Multiplication table of the zygotic algebra for simple Mendelian inheritance

	AA	Aa	aa
AA	AA	$\frac{1}{2}$(AA+Aa)	Aa
Aa	$\frac{1}{2}$(AA+Aa)	$\frac{1}{4}$AA+$\frac{1}{2}$Aa+$\frac{1}{4}$aa	$\frac{1}{2}$(Aa+aa)
aa	Aa	$\frac{1}{2}$(Aa+aa)	aa

The process of constructing a zygotic algebra from the original gametic algebra is called commutative duplication of algebras. We will discuss this process from a mathematical perspective later.

Now that we've seen how the gametic and zygotic algebras are formed in the most basic example, we shall begin to consider the mathematical (and indeed, algebraic) structure of such algebras.

The Nonassociativity of Inheritance

Depending on the "population" you are concerned with, a general element $\alpha A + \beta a$ of the gametic algebra which satisfies $0 \leq \alpha, \beta \leq 1$ and $\alpha + \beta = 1$ can represent a population, a single individual of a population, or a single gamete.

In each case, the coefficients α and β signify the percentage of frequency of the associated allele. I.e., if the element represents a population, then α is the percentage of the population which carries the allele A on the gene under consideration. Likewise, β is the percentage of the population which has the allele a.

For those elements of the gametic and zygotic algebras which represent populations, multiplication of two such elements represents random mating between the two populations.

It seems logical that the order in which populations mate is significant. I.e., if population P mates with population Q and then the resulting population mates with R, the resulting population is not the same as the population resulting from P mating with the population obtained from mating Q and R originally.

Symbolically, $(P \times Q) \times R$ is not equal to $P \times (Q \times R)$.

So, we see that from a purely biological perspective, we should expect that the algebras which arise in genetics will not satisfy the associative property.

Now, if we study the multiplication tables of both the gametic and zygotic algebras for simple Mendelian inheritance, we will notice immediately that the algebras are commutative.

From a biological perspective, if populations P and Q are mating, it makes no difference whether you say P mates with Q or Q mates with P!

However, as we should expect, these algebras do not satisfy the associative property.

E.g., in the gametic algebra apply the rules of multiplication and the distributive property to see that $A \times (A \times a) = \frac{3}{4}A + \frac{1}{4}a$. However,
$(A \times A) \times a = A \times a = \frac{1}{2}A + \frac{1}{2}a$

Hence, the associative property does not hold for the gametic algebra.

The same is true for the zygotic algebra.

In general, the algebras which arise in genetics are commutative but non-associative.

“There is nothing like going to the original sources”

(Ref.—Etherington, “Genetic Algebras,” 1939, §1 pp.242–243,§6,7 249–251)

§1 pp.242–243

“The mechanism of chromosome inheritance, in so far as it determines the probability distributions of genetic types in families and filial generations, and expresses itself through their frequency distributions, may be represented conveniently by algebraic symbols.”

A population (i.e. a distribution of genetic types) is represented by a normalized hypercomplex number in one or another algebra.

If P and Q are populations, the filial generation $P \times Q$ (i.e. the statistical population of offspring resulting from the random mating of individuals of P with individuals of Q) is obtained by multiplying representations of P and Q.

A population may mean a single individual, or rather the information which we may have concerning him in the form of a probability distribution.

Gametic algebras

§6 249–250

Consider the inheritance of traits depending on any number of gene differences at any number of loci on any number of chromosomes in a diploid or generally autopolyploid species.

Let $G_1, \ldots, G_n$ denote the set of gametic types determined by these gene differences. There will be $n(n+1)/2$ zygotic types G_iG_j $(= G_jG_i)$

The formulae giving the series of gametic types by each individual (zygote), and their frequencies may be written $G_iG_j = \gamma_{1ij}G_1 + \cdots + \gamma_{kij}G_k + \cdots + \gamma_{nij}G_n$, with the normalizing conditions $\gamma_{1ij} + \cdots + \gamma_{kij} + \cdots + \gamma_{nij} = 1$ and $0 \leq \gamma_{kij} \leq 1$

γ_{kij} is the probability that an arbitrary gamete produced by an individual of zygotic type G_iG_j is of type G_k

A population P which produces gametes G_k in proportions α_k may be represented by writing $P = \alpha_1 G_1 + \cdots + \alpha_k G_k + \cdots + \alpha_n G_n$, with the normalizing condition $\alpha_1 + \cdots + \alpha_k + \cdots + \alpha_n = 1$

A population may also be described by the proportions of the zygotic types which it contains, so we may write $P = \alpha_{11} G_1 G_1 + \cdots + \alpha_{ij} G_i G_j + \cdots + \alpha_{nn} G_n G_n$, with the normalizing condition $\alpha_{11} + \cdots + \alpha_{ij} + \cdots + \alpha_{nn} = 1 \qquad (\alpha_{ij} = \alpha_{ji})$

If two populations $P = \alpha_1 G_1 + \cdots + \alpha_k G_k + \cdots + \alpha_n G_n$, and $Q = \beta_1 G_1 + \cdots + \beta_k G_k + \cdots + \beta_n G_n$ intermate at random, representations of the first filial generation are obtained by multiplying P and Q

The population of offspring is then

$$PQ = \alpha_1 \beta_1 G_1 G_1 + \cdots + \alpha_i \beta_j G_i G_j + \cdots + \alpha_n \beta_n G_n G_n$$

The linear algebra with basis $G_1, \ldots, G_k, \ldots G_n$ and multiplication table

$$G_i G_j = \gamma_{1ij} G_1 + \cdots + \gamma_{kij} G_k + \cdots + \gamma_{nij} G_n$$

is called the **gametic algebra** for the type of inheritance considered

Zygotic Algebras

§7 250–251

When individuals of type G_iG_j and G_lG_m mate, the probability distribution of zygotic types in their offspring can be obtained by multiplying the gametic representations given by $G_iG_j = \gamma_{1ij}G_1 + \cdots + \gamma_{kij}G_k + \cdots + \gamma_{nij}G_n$ and $G_lG_m = \gamma_{1lm}G_1 + \cdots + \gamma_{klm}G_k + \cdots + \gamma_{nlm}G_n$

We obtain $G_iG_j \times G_lG_m = \gamma_{1ij}\gamma_{1lm}G_1G_1 + \cdots + \gamma_{\sigma ij}\gamma_{\tau lm}G_\sigma G_\tau + \cdots + \gamma_{nij}\gamma_{nlm}G_nG_n$

Or, writing $Z_{ij} = G_iG_j$ to emphasize the union of paired gametes into single individuals,

$$Z_{ij}Z_{lm} = \gamma_{1ij}\gamma_{1lm}Z_{11} + \cdots + \gamma_{\sigma ij}\gamma_{\tau lm}Z_{\sigma\tau} + \cdots + \gamma_{nij}\gamma_{nlm}Z_{nn} \quad (4)$$

The linear algebra with basis $Z_{11}, \ldots, Z_{ij}, \ldots Z_{nn}$ and multiplication table (4) is called the **zygotic algebra** for the type of inheritance considered

(Ref.—Etherington, “Non-associative algebra and the symbolism of genetics,” 1941, §1 p.24, §2 pp.25–26, §5 pp.29–30, §8 pp.34–35)

§1 p.24

“The statistical material of genetics usually consists of frequency distributions—of genes, zygotes and mating couples—from which new distributions referring to their progeny arise.”

“Combination of distributions by random mating is usually symbolized by the mathematical sign for multiplication; but this sign is not taken literally for the simple reason that the general laws connecting the distributions of progenitors and progeny are inconsistent with the laws governing multiplication in algebra.”

“However, there is no insuperable reason why the genetical sign of multiplication should not be taken literally; for it is possible with any particular type of inheritance to construct an ‘algebra’—distinct from ordinary algebra but of a type well known to mathematicians—such that the laws governing multiplication shall represent exactly the underlying genetical situation.”

“These ‘genetic algebras’ are of a kind known as ‘linear algebras.’ ”

Genetical Multiplication

§2 pp.25–26

P denotes a frequency distribution or a probability distribution of a population, a single individual, or a single gamete.

- $P = DD =$ homozygous dominant individual, or population consisting of such
- $P = \alpha DD + \beta DR + \gamma RR =$ population with assigned frequencies α, β, γ of genotypes, or individual with assigned probabilities α, β, γ of belonging to one of the genotypes
- $P = \delta D + \rho R =$ population which produces D and R gametes in the given numerical ratio, or gamete which has probability δ of containing D and probability ρ of containing R

The multiplication of populations (individuals, gametes) means the calculation of progeny distribution resulting from random mating (mating, fusion).

The distributive ($P(Q + R) = PQ + PR$) and commutative ($PQ = QP$) laws are valid in genetical multiplication. The associative law is not: ($P(QR) \neq (PQ)R$).

Mendelian Gametic and Zygotic Algebras—Revisited

§5 pp.29–30

Consider a pair of autosomal allelomorphs D, R and the corresponding genotypes of zygotes $A = DD \quad , \quad B = DR \quad , \quad C = RR$

In accordance with Mendelian principles, we have

- gametes produced by each type of zygote (**gametic algebra**)

$$D^2 = DD = D \quad , \quad DR = \frac{1}{2}D + \frac{1}{2}R \quad , \quad R^2 = RR = R$$

(heterozygote DR produces D and R gametes in equal numbers)

- zygotes produced by each type of mating couple (**zygotic algebra**)

$$A^2 = A \quad , B^2 = \frac{1}{4}A + \frac{1}{2}B + \frac{1}{4}C \quad , \quad C^2 = C$$

$$BC = \frac{1}{2}B + \frac{1}{2}C \quad , \quad CA = B \quad , \quad AB = \frac{1}{2}A + \frac{1}{2}B$$

(the offspring of a mating $DR \times DR$ are 25%DD, 50%DR, 25%RR)

Self Fertilization

§8 pp.34–35 (See also Reed, §5.2 pp.121–122)

Starting from the zygotic distribution $P = \alpha A + \beta B + \gamma C$, where $a = DD, B = DR, C = RR$, if mating proceeds in successive generations by self-fertilization, or by each individual mating with another of the same type, the first filial generation F_1 will consist of the offspring of $A \times A, B \times B, C \times C$, occurring in proportions $\alpha : \beta : \gamma$ so that

$$\begin{aligned} F_1 &= \alpha A^2 + \beta B^2 + \gamma C^2 \\ &= \alpha A + \beta(\frac{1}{4}A + \frac{1}{2}B + \frac{1}{4}C) + \gamma C \\ &= (\alpha + \frac{1}{4}\beta)A + \frac{1}{2}\beta B + (\frac{1}{4}\beta + \gamma)C \end{aligned}$$

The second filial generation $F_2 = F_1 \times F_!$ calculates to

$$F_2 = (\alpha + \frac{3}{8}\beta)A + \frac{1}{4}\beta B + (\frac{3}{8}\beta + \gamma)C$$

QUESTION: What is the n^{th}-filial generation under self-fertilization?

$$F_n = \alpha_n A + \beta_n B + \gamma_n C$$

- $\alpha_1 = \alpha + \frac{1}{4}\beta,\ \alpha_2 = \alpha + \frac{3}{8}\beta$
- $\beta_1 = \frac{1}{2}\beta.\ \beta_2 = \frac{1}{4}\beta$
- $\gamma_1 = \frac{1}{4}\beta + \gamma,\ \gamma_2 = \frac{3}{8}\beta + \gamma$

ANSWER: $\alpha_n = \alpha + \frac{1}{2}\beta - \frac{1}{2^{n+1}}\beta,\qquad \beta_n = \frac{1}{2^n}\beta,\qquad \gamma_n = \frac{1}{2}\beta + \gamma - \frac{1}{2^{n+1}}\beta$

Or, $$F_n = (\alpha + \frac{1}{2}\beta - \frac{1}{2^{n+1}}\beta)A + \frac{1}{2^n}\beta B + (\frac{1}{2}\beta + \gamma - \frac{1}{2^{n+1}}\beta)C$$

The equilibrium distribution is thus

$$F_n = (\alpha + \frac{1}{2}\beta)A + (\frac{1}{2}\beta + \gamma)C$$

Repeated self fertilization kills off the heterozygotes!

Derivations on Linear Algebras

(Ref.—Russo, “Playing havoc with the product rule” Transfer Seminar Fall 2012)

Much of the algebra taught in the undergraduate curriculum, such as linear algebra (**vector spaces, matrices**), modern algebra (**groups, rings, fields**), number theory (**primes, congruences**) is concerned with systems with one or more associative binary products.

For example, addition and multiplication of matrices is associative:

$$A + (B + C) = (A + B) + C$$

$$A(BC) = (AB)C.$$

In the early 20th century, physicists started using the product A.B for matrices, defined by

$$A.B = AB + BA,$$

and called the Jordan product (after the physicist **Pascual Jordan 1902-1980**), to model the observables in quantum mechanics.

Also in the early 20th century both mathematicians and physicists used the product [A,B], defined by

$$[A, B] = AB - BA$$

and called the Lie product (after the mathematician **Sophus Lie 1842-1899**), to study differential equations.

Neither one of these products is associative, so they each give rise to what is called a nonassociative algebra, in these cases, called **Jordan algebras** and **Lie algebras** respectively.

Sophus Lie (1842–1899)

Marius Sophus Lie was a Norwegian mathematician. He largely created the theory of continuous symmetry, and applied it to the study of geometry and differential equations.

Pascual Jordan (1902–1980)

Pascual Jordan was a German theoretical and mathematical physicist who made significant contributions to quantum mechanics and quantum field theory.

Abstract theories of these algebras and other nonassociative algebras were subsequently developed and have many other applications, for example to **cryptography** and **genetics**, to name just two.

Lie algebras are especially important in **particle physics**.

The derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

DIFFERENTIATION IS A LINEAR PROCESS

$$(f+g)' = f' + g'$$

$$(cf)' = cf'$$

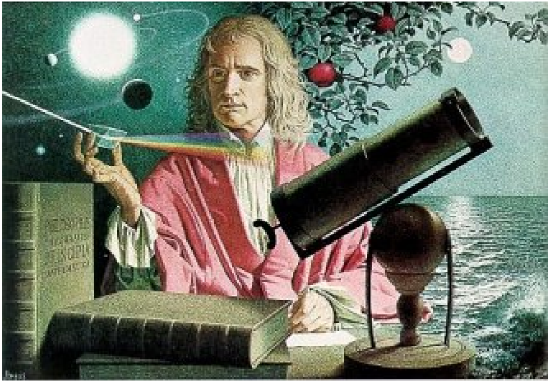
THE SET OF DIFFERENTIABLE FUNCTIONS FORMS AN ALGEBRA $\mathcal{D}$

$$(fg)' = fg' + f'g$$

(product rule)

HEROS OF CALCULUS

#1 Sir Isaac Newton (1642-1727)



Isaac Newton was an English physicist, mathematician, astronomer, natural philosopher, alchemist, and theologian, and is considered by many scholars and members of the general public to be one of the most influential people in human history.

#2 Gottfried Wilhelm Leibniz (1646-1716)

Gottfried Wilhelm Leibniz was a German mathematician and philosopher. He developed the infinitesimal calculus independently of Isaac Newton, and Leibniz's mathematical notation has been widely used ever since it was published.

LEIBNIZ RULE

$$(fg)' = f'g + fg'$$

(order changed)

More generally

$$(fgh)' = f'gh + fg'h + fgh'$$

$$(f_1 f_2 \cdots f_n)' = (f_1' f_2 \cdots f_n) + \cdots + (f_1 \cdots f_i' \cdots f_n) + \cdots + (f_1 f_2 \cdots f_n')$$

The chain rule,

$$(f \circ g)'(x) = f'(g(x))g'(x)$$

plays no role in this seminar. Neither does the quotient rule

$$(f/g)' = \frac{gf' - fg'}{g^2}$$

CONTINUITY: $x_n \to x \Rightarrow f(x_n) \to f(x)$

THE SET OF CONTINUOUS FUNCTIONS FORMS AN ALGEBRA $\mathcal{C}$
(sums, constant multiples and products of continuous functions are continuous)

$\mathcal{D}$ and $\mathcal{C}$ ARE EXAMPLES OF ALGEBRAS WHICH ARE BOTH **ASSOCIATIVE** AND **COMMUTATIVE**

PROPOSITION

EVERY DIFFERENTIABLE FUNCTION IS CONTINUOUS
$\mathcal{D}$ is a subalgebra of $\mathcal{C}$; $\mathcal{D} \subset \mathcal{C}$

DIFFERENTIATION IS A LINEAR PROCESS. LET US DENOTE IT BY D AND WRITE $D(f)$ (or Df) for f'

$$D(f+g) = Df + Dg$$
$$D(cf) = cDf$$
$$D(fg) = (Df)g + f(Dg)$$
$$D(f/g) = \frac{g(Df) - f(Dg)}{g^2}$$

DEFINITION

A DERIVATION ON AN ALGEBRA $\mathcal{A}$ IS A LINEAR PROCESS $\delta : \mathcal{A}$ SATISFYING THE LEIBNIZ RULE:

$$\delta(x+y) = \delta(x) + \delta(y)$$
$$\delta(cx) = c\delta(x)$$
$$\delta(xy) = \delta(x)y + x\delta(y)$$

THEOREM

There are no (non-zero) derivations on $\mathcal{C}$.

In other words, every derivation of $\mathcal{C}$ is identically zero

COROLLARY

$\mathcal{D} \neq \mathcal{C}$

(NO DUUUH! $f(x) = |x|$)

DERIVATIONS ON THE SET OF MATRICES

THE SET $M_n(\mathbb{R})$ of n by n MATRICES IS AN ALGEBRA UNDER
MATRIX ADDITION $A + B$
MATRIX MULTIPLICATION $A \times B$
WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

DEFINITION

A <u>DERIVATION</u> ON $M_n(\mathbb{R})$ WITH <u>RESPECT TO MATRIX MULTIPLICATION</u> IS A LINEAR PROCESS δ WHICH SATISFIES THE LEIBNIZ RULE
$\delta(A \times B) = \delta(A) \times B + A \times \delta(B)$.

EXAMPLE

FIX A MATRIX A in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH CAN BE NON-ZERO)

THEOREM

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbb{R})$.

CLOSING REMARKS (for today)

- If A is any ASSOCIATIVE algebra, and a is any element of A, then the linear process δ_a defined by $\delta_a(x) = ax - xa$ is a derivation of the algebra A.
- In many (but not all) associative algebras, these are the only derivations.
- What about non-associative algebras? In particular, Lie algebras, Jordan algebras, genetic algebras?

THE BRACKET PRODUCT ON MATRICES

DEFINITION THE BRACKET PRODUCT ON THE SET $M_n(\mathbb{R})$ OF MATRICES IS DEFINED BY $[X, Y] = X \times Y - Y \times X$

THE SET $M_n(\mathbb{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND BRACKET MULTIPLICATION, WHICH IS NOT ASSOCIATIVE AND NOT COMMUTATIVE.

A <u>DERIVATION</u> ON $M_n(\mathbb{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE LEIBNIZ RULE $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$

EXAMPLE FIX A MATRIX A in $M_n(\mathbb{R})$ AND DEFINE $\delta_A(X) = [A, X] = A \times X - X \times A$. THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION

THEOREM EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbb{R})$.

THE CIRCLE PRODUCT ON THE SET OF MATRICES

DEFINITION THE CIRCLE PRODUCT ON THE SET $M_n(\mathbb{R})$ OF MATRICES IS DEFINED BY $X \circ Y = (X \times Y + Y \times X)/2$

THE SET $M_n(\mathbb{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND CIRCLE MULTIPLICATION, WHICH IS COMMUTATIVE BUT NOT ASSOCIATIVE.

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE LEIBNIZ RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

EXAMPLE FIX A MATRIX A in $M_n(\mathbb{R})$ AND DEFINE $\delta_A(X) = A \times X - X \times A$. THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION

THEOREM EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbb{R})$.

IT IS TIME FOR A SUMMARY OF THE PRECEDING

Table 6

matrix	bracket	circle
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$

AXIOMATIC APPROACH

AN <u>ALGEBRA</u> IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH TWO BINARY OPERATIONS, CALLED <u>ADDITION</u> AND <u>MULTIPLICATION</u>

ADDITION IS DENOTED BY $a + b$ AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE

$$a + b = b + a, \quad (a + b) + c = a + (b + c)$$

MULTIPLICATION IS DENOTED BY ab AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION

$$(a + b)c = ac + bc, \quad a(b + c) = ab + ac$$

AN ALGEBRA IS SAID TO BE <u>ASSOCIATIVE</u> (RESP. <u>COMMUTATIVE</u>) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE) (RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

THE ALGEBRAS $\mathcal{C}$, $\mathcal{D}$ AND $M_n(\mathbb{R})$ ARE EXAMPLES OF ASSOCIATIVE ALGEBRAS.

$\mathcal{C}$ AND $\mathcal{D}$ ARE COMMUTATIVE, AND $M_n(\mathbb{R})$ IS NOT COMMUTATIVE.

THE AXIOM WHICH CHARACTERIZES ASSOCIATIVE ALGEBRAS IS $a(bc) = (ab)c$. THESE ARE CALLED **ASSOCIATIVE ALGEBRAS**

THE AXIOM WHICH CHARACTERIZES COMMUTATIVE ALGEBRAS IS $ab = ba$. THESE ARE CALLED (you guessed it) **COMMUTATIVE ALGEBRAS**

HOWEVER, THESE TWO CONCEPTS ARE TOO GENERAL TO BE OF ANY USE BY THEMSELVES

THE AXIOMS WHICH CHARACTERIZE BRACKET MULTIPLICATION ARE

$$a^2 = 0 \text{ and } (ab)c + (bc)a + (ca)b = 0$$

THESE ARE CALLED **LIE ALGEBRAS**

THE AXIOMS WHICH CHARACTERIZE CIRCLE MULTIPLICATION ARE

$$ab = ba \text{ and } a(a^2 b) = a^2(ab)$$

THESE ARE CALLED **JORDAN ALGEBRAS**

Sophus Lie (1842–1899)

Marius Sophus Lie was a Norwegian mathematician. He largely created the theory of continuous symmetry, and applied it to the study of geometry and differential equations.

Pascual Jordan (1902–1980)

Pascual Jordan was a German theoretical and mathematical physicist who made significant contributions to quantum mechanics and quantum field theory.

LET'S SUMMARIZE AGAIN

Table 7—ALGEBRAS

commutative algebras
$ab = ba$

associative algebras
$a(bc) = (ab)c$

Lie algebras
$a^2 = 0$
$(ab)c + (bc)a + (ca)b = 0$

Jordan algebras
$ab = ba$
$a(a^2 b) = a^2(ab)$

Closing Remark

Given any algebra A of any kind (associative, Lie, Jordan, genetic, you name it) , the set of all derivations on A is a Lie algebra with the bracket given by

$$[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1$$

For the record, if A is an algebra with product denoted by xy, δ is a derivation if

- $\delta(x + y) = \delta(x) + \delta(y)$
- $\delta(xy) = x\delta y + (\delta x)y$

For any two linear transformations S and T on A, their product ST is defined by

$$ST\ (x) = S(T(x))$$

In particular, $\delta_1\delta_2(x) = \delta_1(\delta_2(x))$

To convince yourself that the remark is true, you have to show that

$$[\delta_1, \delta_2](xy) = x\,([\delta_1, \delta_2](y)) + ([\delta_1, \delta_2](x))\,y$$