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Some remarks on semisimple Leibniz algebras

S. Gómez-Vidal [a], A.Kh. Khudoyberdiyev [b,*], B.A. Omirov [b]

[a] Dpto. Matemática Aplicada I, Universidad de Sevilla, Avda. Reina Mercedes, s/n. 41012 Sevilla, Spain
[b] Institute of Mathematics and Information Technologies of Academy of Uzbekistan, 29, Do'rmon yo'li street, 100125, Tashkent, Uzbekistan

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A B S T R A C T

From Levi's Theorem it is known that every finite-dimensional Lie algebra over a field of characteristic zero is decomposed into semidirect sum of its solvable radical and semisimple subalgebra. Moreover, semisimple part is the direct sum of simple ideals. In Barnes (preprint) [6] Levi's Theorem is extended to the case of Leibniz algebras. In the present paper we investigate the semisimple Leibniz algebras and we show that the splitting theorem for semisimple Leibniz algebras is not true. Moreover, we consider some special classes of the semisimple Leibniz algebras and we find a condition under which they decompose into direct sum of simple ideals.

1. Introduction

Due to Levi's Theorem the study of finite-dimensional Lie algebras over a field of characteristic zero is reduced to the study of solvable and semisimple algebras [7]. From results of [12] we can conclude that the main part of solvable Lie algebra consists of the maximal nilpotent ideal. The classification of semisimple Lie algebras has been known since the works of Cartan and Killing [7]. According to the Cartan–Killing theory the semisimple Lie algebra can be represented as a direct sum of simple Lie algebras.

* Corresponding author.
E-mail addresses: samuel.gomezvidal@gmail.com (S. Gómez-Vidal), khabror@mail.ru (A.Kh. Khudoyberdiyev), omirovb@mail.ru (B.A. Omirov).

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Recall that the variety of algebras defined by fundamental identity

$$\big[x,[y,z]\big]=\big[[x,y],z\big]-\big[[x,z],y\big]$$

is a non-antisymmetric generalization of Lie algebras and they have been firstly considered in 1965 by Bloh [5], who called them D-algebras. Unfortunately, his work did not give impulse for study of such algebras. Later on at the beginning of 90-th of the last century the same variety of algebras appear in the work of Loday [8] by terminology of Leibniz algebras.

The last 20 years the theory of Leibniz algebras has been actively studied and many results of the theory of Lie algebras have been extended to Leibniz algebras. Until now a lot of works are devoted to the description of finite-dimensional nilpotent Leibniz algebras [2–4]. However, simple and semisimple parts are not studied. It is because the notions of simple and semisimple Leibniz algebras do not agree with the corresponding classical notions. In fact, in non-Lie Leibniz algebra L there is non-trivial ideal, which is a subspace spanned by squares of elements of the algebra L (denoted by I). Therefore, in [1] the notion of simple Leibniz algebra has been suggested, namely, a Leibniz algebra L is called *simple* if it contains only ideals $\{0\}$, I, L and square of the algebra is not equal to the ideal I. In the case when the Leibniz algebra is Lie algebra, the ideal I is trivial and this definition agrees with the definition of simple Lie algebra. Obviously, the quotient algebra by ideal I of simple Leibniz algebra is simple Lie algebra, but the converse is not true.

From an analogue of Levi's Theorem for Leibniz algebras [6] the description of simple Leibniz algebras immediately follows. In the present paper we present the same description but with another proof. Moreover, we introduce a notion of a semisimple Leibniz algebra (algebra whose solvable radical is coincided with I) and investigate such algebras. Note that Leibniz algebra is semisimple if and only if quotient Lie algebra is semisimple. In particular, we find some sufficient conditions under which an analogue of splitting theorem for semisimple Leibniz algebras is true. In addition, an example of semisimple Leibniz algebra, which is not decomposed into a direct sum of simple Leibniz ideals, is given.

Actually, there exist semisimple Leibniz algebras (which are not simple in general) for which the quotient algebra is simple Lie algebra. So, we call such algebras *Lie-simple* Leibniz algebras. According to this definition the natural question arises – whether an arbitrary finite-dimensional semisimple Leibniz algebra is a direct sum of Lie-simple Leibniz algebras. We show that the answer to the question is also negative, we give a counterexample.

Finally, for some special classes of semisimple Leibniz algebras we give sufficient conditions under which these classes decomposed into a direct sum of the Lie-simple Leibniz algebras. More precisely, we consider a semisimple Leibniz algebra consisting of the direct sum of the classical three-dimensional simple Lie algebras sl_2.

In this paper all algebras and vector spaces are considered over a field of characteristic zero and finite-dimensional.

We shall use symbols: $+$, $\oplus$ and $\dot{+}$ for notations of direct sum of vector spaces, direct and semidirect sums of algebras, respectively.

2. Preliminaries

In this section we give necessary definitions and preliminary results.

Definition 2.1. An algebra $(L,[\cdot,\cdot])$ over a field F is called a Leibniz algebra if for any $x,y,z\in L$ the so-called Leibniz identity

$$\big[x,[y,z]\big]=\big[[x,y],z\big]-\big[[x,z],y\big]$$

holds true.

For a given Leibniz algebra L we define derived sequence as follows:

$$L^1 = L, \qquad L^{[n+1]} = \left[L^{[n]}, L^{[n]}\right], \quad n \geqslant 1.$$

Definition 2.2. A Leibniz algebra L is called solvable, if there exists $m \in \mathbb{N}$ such that $L^{[m]} = 0$. The minimal number m with this property is called index of solvability of the algebra L.

Let us recall Levi's Theorem for Lie algebras.

Theorem 2.3. *(See [7].) For an arbitrary finite-dimensional Lie algebra B over a field of characteristic zero with solvable radical R, there exists semisimple subalgebra G such that $B = G \dotplus R$.*

Further we shall need the following splitting theorem for semisimple Lie algebras.

Theorem 2.4. *(See [7].) An arbitrary finite-dimensional semisimple Lie algebra is decomposed into a direct sum of simple ideals and the decomposition is unique up to permutations of summands.*

In [6] it is shown that for Leibniz algebra L the ideal $I = id\langle [x, x] \mid x \in L\rangle$ coincides with the space spanned by squares of elements of L. Moreover, it is readily to see that this ideal belongs to right annihilator, that is $[L, I] = 0$. Note that the ideal I is the minimal ideal with respect to the property that the quotient algebra L/I is a Lie algebra.

According to [7] a three-dimensional simple Lie algebra is said to be *split* if the algebra contains an element h such that $ad\,h$ has a non-zero characteristic root ρ belonging to the base field. It is well known that any such algebra has a basis $\{e, f, h\}$ with the multiplication table

$$\begin{aligned} &[e, h] = 2e, \qquad [f, h] = -2f, \qquad [e, f] = h, \\ &[h, e] = -2e, \qquad [h, f] = 2f, \qquad [f, e] = -h. \end{aligned}$$

This simple 3-dimensional Lie algebra denoted by sl_2 and the basis $\{e, f, h\}$ is called *canonical basis*. Note that any 3-dimensional simple Lie algebra is isomorphic to sl_2.

Here is the result of [11] which describes simple Leibniz algebras whose quotient Lie algebras are isomorphic to sl_2.

Theorem 2.5. *Let L be a complex finite-dimensional simple Leibniz algebra. Assume that the quotient Lie algebra L/I is isomorphic to the algebra sl_2. Then there exists a basis $\{e, f, h, x_0, x_1, \dots, x_m\}$ of L such that non-zero products of basis elements in L are represented as follows:*

$$\begin{aligned} &[e, h] = 2e, && [h, f] = 2f, && [e, f] = h, \\ &[h, e] = -2e, && [f, h] = -2f, && [f, e] = -h, \\ &[x_k, h] = (m - 2k)x_k, && 0 \leqslant k \leqslant m, && \\ &[x_k, f] = x_{k+1}, && 0 \leqslant k \leqslant m - 1, && \\ &[x_k, e] = -k(m + 1 - k)x_{k-1}, && 1 \leqslant k \leqslant m. && \end{aligned}$$

In [11] Leibniz algebras (they are not necessary to be simple) for which the quotient Lie algebras are isomorphic to sl_2 are described. Let us present a Leibniz algebra L with table of multiplication in a basis $\{e, f, h, x_1^j, \dots, x_{t_j}^j,\ 1 \leqslant j \leqslant p\}$ which is not simple, but the quotient algebra L/I is a simple [11]:

$$\begin{array}{lll}
[e,h]=2e, & [h,f]=2f, & [e,f]=h, \\
[h,e]=-2e, & [f,h]=-2f, & [f,e]=-h, \\
\left[x_k^j,h\right]=(t_j-2k)x_k^j, & 0\leqslant k\leqslant t_j, & \\
\left[x_k^j,f\right]=x_{k+1}^j, & 0\leqslant k\leqslant t_j-1, & \\
\left[x_k^j,e\right]=-k(t_j+1-k)x_{k-1}^j, & 1\leqslant k\leqslant t_j, &
\end{array}$$

where $L=sl_2+I_1+I_2+\cdots+I_p$ and $I_j=\langle x_1^j,\ldots,x_{t_j}^j\rangle,\ 1\leqslant j\leqslant p$.

Note that the Leibniz algebras presented above are examples of non-simple but Lie-simple Leibniz algebras.

Now we introduce a notion of semisimplicity for Leibniz algebras.

Definition 2.6. A Leibniz algebra L is called semisimple if its maximal solvable ideal is equal to I.

Since in Lie algebras case the ideal I is equal to zero, this definition also agrees with the definition of semisimple Lie algebra.

Obviously, for the sets of n-dimensional simple ($SimpL_n$), Lie-simple ($LieSimpL_n$) and semisimple ($SemiSimpL_n$) Leibniz algebras the following embeddings are true:

$$SimpL_n \subseteq LieSimpL_n \subseteq SemiSimpL_n.$$

Although Levi's Theorem is proved for the left Leibniz algebras [6], it is also true for right Leibniz algebras (here we considering right Leibniz algebras).

Theorem 2.7 *(Levi's Theorem). (See [6].) Let L be a finite-dimensional Leibniz algebra over a field of characteristic zero and R be its solvable radical. Then there exists a semisimple subalgebra S of L, such that $L=S\dotplus R$.*

From the proof of Theorem 2.7 it is not difficult to see that S is a semisimple Lie algebra. Therefore, we have that a simple Leibniz algebra is a semidirect sum of simple Lie algebra S and irreducible right module I, i.e. $L=S\dotplus I$. Hence, we get the description of simple Leibniz algebras in terms of simple Lie algebras and their ideals I. For example see the algebras of Theorem 2.5.

Definition 2.8. A non-zero module M whose only submodules are the module itself and zero module is called *irreducible module*. A non-zero module M which is a direct sum of irreducible modules is said to be *completely reducible*.

Further we shall use the following classical result of the theory of Lie algebras.

Theorem 2.9. *(See [7].) Let G be a semisimple Lie algebra over a field of characteristic zero. Then every finite-dimensional module over G is completely reducible.*

Here is an example of simple Leibniz algebras constructed in [1].

Example 1. Let G be a simple Lie algebra and M be an irreducible skew-symmetric G-module (i.e. $[x,m]=0$ for all $x\in G$, $m\in M$). Then the vector space $Q=G+M$ equipped with the multiplication $[x+m,y+n]=[x,y]+[m,y]$, where $m,n\in M$, $x,y\in G$ is a simple Leibniz algebra.

3. The main results

As it was mentioned above from Theorem 2.7 it follows that any simple Leibniz algebra is presented as a semidirect sum of a simple Lie algebra and the ideal I.

Below we give another proof of the description of simple Leibniz algebras without using Levi's Theorem.

Theorem 3.1. *Let L be a finite-dimensional simple Leibniz algebra. Then it has the construction of Example 1 for $G \cong L/I$ and $M = I$.*

Proof. Let L be an algebra satisfying the conditions of theorem. It should be noted, that the ideal I may be considered as a right L/I-module by the action:

$$m * (a + I) = [m, a],$$

where $m \in I$, $a + I \in L/I$. Since L/I is a simple Lie algebra then by Whytehead's Lemma [7] we have $H^2(L/I, I) = 0$.

J.-L. Loday and T. Pirashvili [9] established that there exist the following natural bijection:

$$Ext(L, M) \cong HL^2(L, M).$$

Recall a result of T. Pirashvili [10], which says the following:

Let g be a semisimple Lie algebra and M a right irreducible module over g such that $H^2(g, M) = 0$. Then $HL^2(g, M) = 0$.

Let now $L = L' \oplus I$ is a direct sum vector spaces, where $L' \cong L/I$ is a simple Lie algebra. Since ideal I is contained in right annihilator of the algebra L then we have $[L', I] = 0$ and $[I, I] = 0$. Due to simplicity of Leibniz algebra L we derive that the ideal I is irreducible L/I-module. Using Pirashvili's result we have that $HL^2(L/I, I) = 0$, but the condition $0 = HL^2(L/I, I) \cong Ext(L/I, I)$ is equivalent to $L \cong L/I \dot{+} I$. $\square$

Let us investigate the case of semisimple Leibniz algebras. Let L be a semisimple Leibniz algebra. Similarly to the case of simple Leibniz algebras we can establish that $L \cong L/I \dot{+} I$. It is known the result on decomposition of semisimple Lie algebra L/I into a direct sum of simple Lie ideals. Moreover, we have that L/I-module I is completely reducible and hence, ideal I is decomposed into a direct sum of irreducible submodules over the Lie algebra L/I.

Taking into account these results for semisimple Leibniz algebras it seems that the following conclusion is true for Leibniz algebras case:

Conjecture. *An arbitrary finite-dimensional semisimple Leibniz algebra is decomposed into direct sum of simple Leibniz algebras.*

Let L be a finite-dimensional semisimple Leibniz algebra. Then according to Theorem 2.7 we have that $L = S \dot{+} I$, where S is a semisimple Lie algebra and $[I, S] = I$. From Theorem 2.4 we get $S = S_1 \oplus S_2 \oplus \cdots \oplus S_k$, where S_i $(1 \leqslant i \leqslant k)$ is a simple Lie algebra. Thus, we have

$$L = (S_1 \oplus S_2 \oplus \cdots \oplus S_k) \dot{+} I.$$

Let us introduce the denotation $I_j = [I, S_j]$ for $1 \leqslant j \leqslant k$.

Lemma 3.2. *The following are true*:

a) $I = I_1 + I_2 + \cdots + I_k$;
b) *I_j is an ideal of L for all j $(1 \leqslant j \leqslant k)$*;

c) $I_j = [I_j, S_j]$ *for all* j $(1 \leqslant j \leqslant k)$;
d) $S_j + I_j$ *is an ideal of* L *for all* j $(1 \leqslant j \leqslant k)$.

Proof. Since I is an ideal of L then $I_j = [I, S_j] \subseteq I$ for all j. Hence, $I_1 + I_2 + \cdots + I_k \subseteq I$.
From

$$I = [I, S] = [I, S_1 \oplus S_2 \oplus \cdots \oplus S_k] \subseteq [I, S_1] + [I, S_2] + \cdots + [I, S_k] = I_1 + I_2 + \cdots + I_k,$$

we have the correctness of the statement a).

The proof of the statement b) follows from property $[S_j, S_j] = S_j$ and we have

$$\begin{aligned}
[I_j, L] &= \big[[I, S_j], L\big] \subseteq \big[I, [S_j, L]\big] + \big[[I, L], S_j\big] \\
&= \big[I, \big[S_j, (S_1 \oplus S_2 \oplus \cdots \oplus S_k) \dot{+} I\big]\big] + \big[\big[I, (S_1 \oplus S_2 \oplus \cdots \oplus S_k) \dot{+} I\big], S_j\big] \\
&\subseteq \big[I, [S_j, S_1 \oplus S_2 \oplus \cdots \oplus S_k]\big] + \big[[I, S_1 \oplus S_2 \oplus \cdots \oplus S_k], S_j\big] \\
&\subseteq \big[I, [S_j, S_j]\big] + [I, S_j] \subseteq [I, S_j] = I_j.
\end{aligned}$$

Since from b) we have that I_j is an ideal of L, we obtain $[I_j, S_j] \subseteq I_j$ and from

$$I_j = [I, S_j] = \big[I, [S_j, S_j]\big] \subseteq \big[[I, S_j], S_j\big] + \big[[I, S_j], S_j\big] = [I_j, S_j],$$

we get $[I_j, S_j] = I_j$. So, the statement c) is also proved.

From the following equalities:

$$\begin{aligned}
[S_j + I_j, L] &= \big[S_j + I_j, (S_1 \oplus S_2 \oplus \cdots \oplus S_k) \dot{+} I\big] \\
&= [S_j, S_1 \oplus S_2 \oplus \cdots \oplus S_k] + [S_j, I] + [I_j, S_1 \oplus S_2 \oplus \cdots \oplus S_k] + [I_j, I] \subseteq S_j + I_j, \\
[L, S_j + I_j] &= \big[(S_1 \oplus S_2 \oplus \cdots \oplus S_k) \dot{+} I, S_j + I_j\big] \\
&= [S_1 \oplus S_2 \oplus \cdots \oplus S_k, S_j] + [I, S_j] + [S_1 \oplus S_2 \oplus \cdots \oplus S_k, I_j] + [I, I_j] \subseteq S_j + I_j,
\end{aligned}$$

we get that $S_j + I_j$ is an ideal of L. Thus, the part d) is also proved. □

The following example shows that the Conjecture is not true in general.

Example 2. Let L be a semisimple Leibniz algebra such that $L = (sl_2^1 \oplus sl_2^2) \dot{+} I$, where $I = I_1 \oplus I_2$ and $[I_1, sl_2^2] = [I_2, sl_2^1] = 0$. Moreover, $I_1 = I_{1,1} \oplus I_{1,2}$, $I_2 = I_{2,1} \oplus I_{2,2}$, where $I_{1,1}$ and $I_{1,2}$ are irreducible sl_2^1-modules. Respectively, $I_{2,1}$ and $I_{2,2}$ are irreducible sl_2^2-modules. Then, using Theorem 2.5 we conclude that, there exists a basis $\{e_1, h_1, f_1, e_2, h_2, f_2, x_0^j, x_1^j, \ldots, x_{t_j}^j\}$ $(1 \leqslant j \leqslant 4)$ such that multiplication table of L in this basis has the following form:

$$\big[sl_2^i, sl_2^i\big]: \quad \begin{array}{llll} [e_i, h_i] = 2e_i, & [f_i, h_i] = -2f_i, & [e_i, f_i] = h_i, & \\ {[h_i, e_i]} = -2e_i & [h_i, f_i] = 2f_i, & [f_i, e_i] = -h_i, & i = 1, 2, \end{array}$$

$$\big[I_1, sl_2^1\big]: \quad \begin{array}{ll} \big[x_k^j, h_1\big] = (t_j - 2k)x_k^j, & 0 \leqslant k \leqslant t_j, \\ \big[x_k^j, f_1\big] = x_{k+1}^j, & 0 \leqslant k \leqslant t_j - 1, \\ \big[x_k^j, e_1\big] = -k(t_j + 1 - k)x_{k-1}^j, & 1 \leqslant k \leqslant t_j,\ j = 1, 2, \end{array}$$

$$[I_2, sl_2^2]: \quad \begin{array}{ll} [x_k^j, h_2] = (t_j - 2k)x_k^j, & 0 \leqslant k \leqslant t_j, \\ [x_k^j, f_2] = x_{k+1}^j, & 0 \leqslant k \leqslant t_j - 1, \\ [x_k^j, e_2] = -k(t_j + 1 - k)x_{k-1}^j, & 1 \leqslant k \leqslant t_j, \ j = 3, 4, \end{array}$$

where $I_{1,1} = \{x_0^1, \dots, x_{t_1}^1\}$, $I_{1,2} = \{x_0^2, \dots, x_{t_2}^2\}$, $I_{2,1} = \{x_0^3, \dots, x_{t_3}^3\}$ and $I_{2,2} = \{x_0^4, \dots, x_{t_4}^4\}$.

Evidently, the algebra L is semisimple and it is decomposed into the direct sum of two ideals $sl_2^1 \dot{+} (I_{1,1} \oplus I_{1,2})$ and $sl_2^2 \dot{+} (I_{2,1} \oplus I_{2,2})$, which are not simple Leibniz algebras, but they are not simple Leibniz algebras (they are Lie-simple Leibniz algebras).

Example 2 shows that if I_j is a reducible module over a simple Lie algebra S_j, then Conjecture is not true. Now we consider the case of I_j is an irreducible module. First we prove the following lemma.

Lemma 3.3. *Let L be a semisimple Leibniz algebra such that $L = (sl_2 \oplus S) \dot{+} I$, where S is an arbitrary simple Lie algebra. Let I is irreducible over sl_2, then $[I, S] = 0$.*

Proof. Let $dim\, I = m + 1$, then similarly as in the proof of Theorem 2.5 we have the existence of basis $\{e, f, h, x_0, x_1, \dots, x_m\}$ of $sl_2 \dot{+} I$ such that table of multiplication has the following form:

$$\begin{array}{lll} [e, h] = 2e, & [h, f] = 2f, & [e, f] = h, \\ [h, e] = -2e, & [f, h] = -2f, & [f, e] = -h, \\ [x_i, h] = (m - 2i)x_i, & 0 \leqslant i \leqslant m, & \\ [x_i, f] = x_{i+1}, & 0 \leqslant i \leqslant m - 1, & \\ [x_i, e] = -i(m + 1 - i)x_{i-1}, & 1 \leqslant i \leqslant m. & \end{array}$$

Let $\{y_1, y_2, \dots, y_n\}$ be a basis of the algebra S. We set

$$[x_0, y_j] = \sum_{k=0}^{m} \alpha_{k,j} x_k, \quad 1 \leqslant j \leqslant n.$$

Consider the Leibniz identity

$$\big[[x_0, y_j], f\big] = \big[x_0, [y_j, f]\big] + \big[[x_0, f], y_j\big] = [x_1, y_j].$$

On the other hand

$$\big[[x_0, y_j], f\big] = \left[\sum_{k=0}^{m} \alpha_{k,j} x_k, f\right] = \sum_{k=0}^{m-1} \alpha_{k,j} x_{k+1}.$$

Hence, we get

$$[x_1, y_j] = \sum_{k=0}^{m-1} \alpha_{k,j} x_{k+1}, \quad 1 \leqslant j \leqslant n.$$

Applying induction process and the equality $[[x_i, y_j], f] = [x_i, [y_j, f]] + [[x_i, f], y_j]$, we obtain

$$[x_i, y_j] = \sum_{k=0}^{m-i} \alpha_{k,j} x_{k+i}, \quad 0 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

Consider the Leibniz identity

$$\big[[x_0, y_j], e\big] = \big[x_0, [y_j, e]\big] + \big[[x_0, e], y_j\big] = 0.$$

On the other hand,

$$\big[[x_0, y_j], e\big] = \left[\sum_{k=0}^{m} \alpha_{k,j} x_k, e\right] = \sum_{k=1}^{m} k(-m-1+k)\alpha_{k,j} x_{k-1}.$$

We obtain $\alpha_{i,j} = 0$, for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. For the sake of convenience we shall assume $\alpha_j := \alpha_{0,j}$, i.e. we have

$$[x_i, y_j] = \alpha_j x_i, \quad 0 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

Using the Leibniz identity, we have

$$\big[x_i, [y_j, y_k]\big] = \big[[x_i, y_j], y_k\big] - \big[[x_i, y_k], y_j\big] = [\alpha_j x_i, y_k] - [\alpha_k x_i, y_j] = \alpha_j \alpha_k x_i - \alpha_k \alpha_j x_i = 0.$$

Taking into account the property $[S, S] = S$ and arbitrariness of elements $\{x_i, y_j, y_k\}$ we get $[I, [S, S]] = [I, S] = 0$. $\square$

In the following theorem we show the trueness of the Conjecture under some conditions.

Theorem 3.4. *Let L be a semisimple Leibniz algebras such that $L = (sl_2^1 \oplus sl_2^2 \oplus \cdots \oplus sl_2^{k-1} \oplus S_k) \dot{+} I$. Let I_j is irreducible module over sl_2^j for $j = 1, \dots, k-1$ and I_k is irreducible over S_k. Then L is decomposed into direct sum of simple Leibniz algebras, namely,*

$$L = \big(sl_2^1 \dot{+} I_1\big) \oplus \big(sl_2^2 \dot{+} I_2\big) \oplus \cdots \oplus \big(sl_2^{k-1} \dot{+} I_{k-1}\big) \oplus (S_k \dot{+} I_k).$$

Proof. By Lemma 3.2 it is known that $sl_2^j + I_j$ and $S_k + I_k$ are ideals of L. Since I_j is irreducible over sl_2^j, then by Lemma 3.3 we obtain $[I_j, sl_2^i] = 0$ for all i, j $(i \neq j)$.

Taking into account that $I_j \subseteq I$ and $[L, I] = 0$ we conclude

$$\big[sl_2^i + I_i, sl_2^j + I_j\big] = 0, \quad i \neq j,\ 1 \leqslant i, j \leqslant k-1.$$

Moreover, from Lemma 3.3 we have $[I_i, S_k + I_k] = 0$ for $1 \leqslant i \leqslant k-1$. In order to complete the proof of theorem it is necessary to establish the equality $[I_k, sl_2^j] = 0$ for all $j = 1, \dots, k-1$.

Let us assume the contrary, i.e. $[I_k, sl_2^j] \neq 0$, for some j $(1 \leqslant j \leqslant k-1)$. Since I_k is an ideal of L, then we have $[I_k, sl_2^j] \subseteq I_k$.

From

$$\big[I_k, sl_2^j\big] = \big[[I, S_k], sl_2^j\big] \subseteq \big[I, \big[S_k, sl_2^j\big]\big] + \big[\big[I, sl_2^j\big], S_k\big] = [I_j, S_k] \subseteq I_j,$$

we obtain $[I_k, sl_2^j] \subseteq I_j$.

Hence, we get $I_k \cap I_j \neq 0$. Since $I_k \cap I_j$ is an ideal of the algebra L, then it can be considered as the right module over sl_2^j and S_k. Due to I_j and I_k are irreducible, we have that $I_k \cap I_j = I_j = I_k$. From $[I_j, S_k] = 0$ we derive $[I_k, S_k] = 0$, but it is a contradiction with condition

$$[I_k, S_k] = I_k.$$

Therefore, we have

$$\left[I_k, sl_2^j\right] = 0.$$

Thus, we get that $[S_k + I_k, sl_2^j + I_j] = [sl_2^j + I_j, S_k + I_k] = 0$, which leads that the Leibniz algebra L is decomposed into direct sum of simple ideals. $\square$

In Example 2, it is shown that if I_j is reducible over S_j, then the semisimple Leibniz algebra is not decomposable into direct sum of simple ideals. However this algebra is decomposed into direct sum of Lie-simple algebras.

Then a natural question arises: whether any semisimple Leibniz algebra can be represented as a direct sum of Lie-simple Leibniz algebras.

The following example gives the negative answer to this question.

Example 3. Let L be a 10-dimensional semisimple Leibniz algebra. Let $\{e_1, h_1, f_1, e_2, h_2, f_2, x_1, x_2, x_3, x_4\}$ be a basis of the algebra L such that $I = \{x_1, x_2, x_3, x_4\}$, and multiplication table of L has the following form:

$$\left[sl_2^i, sl_2^i\right]: \quad \begin{array}{lll} [e_i, h_i] = 2e_i, & [f_i, h_i] = -2f_i, & [e_i, f_i] = h_i, \\ [h_i, e_i] = -2e_i & [h_i, f_i] = 2f_i, & [f_i, e_i] = -h_i, \quad i = 1, 2, \end{array}$$

$$\left[I, sl_2^1\right]: \quad \begin{array}{llll} [x_1, f_1] = x_2, & [x_1, h_1] = x_1, & [x_2, e_1] = -x_1, & [x_2, h_1] = -x_2, \\ [x_3, f_1] = x_4, & [x_3, h_1] = x_3, & [x_4, e_1] = -x_3, & [x_4, h_1] = -x_4, \end{array}$$

$$\left[I, sl_2^2\right]: \quad \begin{array}{llll} [x_1, f_2] = x_3, & [x_1, h_2] = x_1, & [x_3, e_2] = -x_1, & [x_3, h_2] = -x_3, \\ [x_2, f_2] = x_4, & [x_2, h_2] = x_2, & [x_4, e_2] = -x_2, & [x_4, h_2] = -x_4 \end{array}$$

(omitted products are equal to zero).

From this table of multiplications we have $[I, sl_2^1] = [I, sl_2^2] = I$. Moreover, I splits over sl_2^1 (i.e. $I = \langle x_1, x_2\rangle \oplus \langle x_3, x_4\rangle$) and over sl_2^2 (i.e. $I = \langle x_1, x_3\rangle \oplus \langle x_2, x_4\rangle$). Therefore,

$$L = \left(sl_2^1 \oplus sl_2^2\right) \dotplus I \neq \left(sl_2^1 \dotplus I_1\right) \oplus \left(sl_2^2 \dotplus I_2\right).$$

Indeed, if the equality $(sl_2^1 \oplus sl_2^2) \dotplus I = (sl_2^3 \dotplus I_1) \oplus (sl_2^4 \dotplus I_2)$ holds for some copies of sl_2^3, sl_2^4 and I_1, I_2, then for dimensions of (I_1, I_2) we need to consider the cases: $(4, 0)$, $(3, 1)$ and $(2, 2)$.

If $(dim\, I_1, dim\, I_2) = (4, 0)$, then $[I_1, sl_2^4] = 0$. From the condition $[I_1, a] = 0$ for an element $a \in sl_2^4$ we deduce $a \in I$, which is a contradiction.

If $(dim\, I_1, dim\, I_2) = (3, 1)$, then $0 \neq I_2$ belongs to $Center(L)$, but $Center(L) = 0$. So, we have a contradiction.

If $(dim\, I_1, dim\, I_2) = (2, 2)$, then we put $I_1 = \langle y_1, y_2\rangle$, $I_2 = \langle y_3, y_4\rangle$ and $y_1 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4$. Taking into account that I_1 is an ideal of L, the products

$$[y_1, e_1],\ [y_1, f_1],\ [y_1, h_1],\ [y_1, e_2],\ [y_1, f_2],\ [y_1, h_2]$$

belong to I_1 and therefore,

$$rank \begin{bmatrix} \alpha_2 & 0 & \alpha_4 & 0 \\ 0 & \alpha_1 & 0 & \alpha_3 \\ \alpha_1 & -\alpha_2 & \alpha_3 & -\alpha_4 \\ \alpha_3 & \alpha_4 & 0 & 0 \\ 0 & 0 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_2 & -\alpha_3 & -\alpha_4 \end{bmatrix} \leqslant 2.$$

This condition implies $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, which is a contradiction. Therefore, the equality $(sl_2^1 \oplus sl_2^2) \dot{+} I = (sl_2^3 \dot{+} I_1) \oplus (sl_2^4 \dot{+} I_2)$ is impossible.

Below we find some types of semisimple Leibniz algebras which are decomposed into a direct sum of Lie-simple Leibniz algebras.

Let L be a semisimple Leibniz algebra such that $L = (sl_2 \oplus S) \dot{+} I$, where S is a simple Lie algebra. Let $I_1 = [I, sl_2]$ be irreducible over sl_2, then according to Theorem 2.9 the module I_1 is completely reducible, i.e. $I_1 = I_{1,1} \oplus I_{1,2} \oplus \cdots \oplus I_{1,p}$, where $I_{1,i}$ is irreducible over sl_2.

Let us consider the case of $p = 2$.

Proposition 3.5. *If* $\dim I_{1,1} \neq \dim I_{1,2}$, *then* $L = (sl_2 \dot{+} I_1) \oplus (S \dot{+} I_2)$.

Proof. Let $I_1 = I_{1,1} \oplus I_{1,2}$, then there exists a basis $\{e, h, f, x_0^1, x_1^1, \dots, x_{t_1}^1, x_0^2, x_1^2, \dots, x_{t_2}^2\}$ of $sl_2 + I_1$, such that

$$[I_1, sl_2]: \quad \begin{aligned} &[x_k^j, h] = (t_j - 2k)x_k^j, && 0 \leqslant k \leqslant t_j, \\ &[x_k^j, f] = x_{k+1}^j, && 0 \leqslant k \leqslant t_j - 1, \\ &[x_k^j, e] = -k(t_j + 1 - k)x_{k-1}^j, && 1 \leqslant k \leqslant t_j,\ j = 1, 2, \end{aligned}$$

where $I_{1,1} = \langle x_0^1, \dots, x_{t_1}^1 \rangle$, $I_{1,2} = \langle x_0^2, \dots, x_{t_2}^2 \rangle$.

Without loss of generality we can assume that $t_1 > t_2$.

Let $\{y_1, y_2, \dots, y_m\}$ be a basis of the algebra S.

We put

$$[x_0^i, y_1] = \sum_{j=1}^{2} \sum_{r=0}^{t_j} \alpha_{i,r}^j x_r^j, \quad 1 \leqslant i \leqslant 2.$$

Consider equalities

$$[[x_0^1, f], y_1] = [x_0^1, [f, y_1]] + [[x_0^1, y_1], f] = \left[\sum_{j=1}^{2} \sum_{r=0}^{t_j} \alpha_{1,r}^j x_r^j, f \right] = \sum_{j=1}^{2} \sum_{r=0}^{t_j - 1} \alpha_{1,r}^j x_{r+1}^j.$$

On the other hand

$$[[x_0^1, f], y_1] = [x_1^1, y_1].$$

Hence, we get

$$[x_1^1, y_1] = \sum_{j=1}^{2} \sum_{r=0}^{t_j - 1} \alpha_{1,r}^j x_{r+1}^j.$$

From equalities

$$\left[x_k^1, y_1\right] = \left[\left[x_{k-1}^1, f\right], y_1\right] = \left[x_{k-1}^1, [f, y_1]\right] + \left[\left[x_{k-1}^1, y_1\right], f\right]$$

and induction we derive

$$\left[x_k^1, y_1\right] = \sum_{j=1}^{2} \sum_{r=0}^{t_j-k} \alpha_{1,r}^j x_{r+k}^j, \quad 1 \leqslant k \leqslant t_2,$$

$$\left[x_k^1, y_1\right] = \sum_{r=0}^{t_1-k} \alpha_{1,r}^j x_{r+k}^j, \quad t_2 + 1 \leqslant k \leqslant t_1.$$

Consider the products

$$\begin{aligned}\left[\left[x_0^1, e\right], y_1\right] &= \left[x_0^1, [e, y_1]\right] + \left[\left[x_0^1, y_1\right], e\right] = \left[\left[x_0^1, y_1\right], e\right] \\ &= \left[\sum_{j=1}^{2} \sum_{r=0}^{t_j} \alpha_{1,r}^j x_r^j, e\right] = \sum_{j=1}^{2} \sum_{r=1}^{t_j} \alpha_{1,r}^j \big(-r(t_j + 1 - r)\big) x_{r-1}^j.\end{aligned}$$

On the other hand

$$\left[\left[x_0^1, e\right], y_1\right] = 0.$$

Comparing the coefficients at the basis elements, we obtain

$$\alpha_{1,r}^j = 0, \quad 1 \leqslant j \leqslant 2,\ 1 \leqslant r \leqslant t_j.$$

Thus, we have

$$\left[x_k^1, y_1\right] = \alpha_{1,0}^1 x_k^1 + \alpha_{1,0}^2 x_k^2, \quad 0 \leqslant k \leqslant t_2,$$

$$\left[x_k^1, y_1\right] = \alpha_{1,0}^1 x_k^1, \quad t_2 + 1 \leqslant k \leqslant t_1.$$

Similarly, from the products $[[x_k^2, f], y_1]$ for $0 \leqslant k \leqslant t_2 - 1$ and $[[x_0^2, e], y_1]$ we obtain

$$\left[x_k^2, y_1\right] = \alpha_{2,0}^1 x_k^1 + \alpha_{2,0}^2 x_k^2, \quad 0 \leqslant k \leqslant t_2.$$

Consider

$$\left[\left[x_{t_2}^2, f\right], y_1\right] = \left[x_{t_2}^2, [f, y_1]\right] + \left[\left[x_{t_2}^2, y_1\right], f\right] = \left[\alpha_{2,0}^1 x_{t_2}^1 + \alpha_{2,0}^2 x_{t_2}^2, f\right] = \alpha_{2,0}^1 x_{t_2+1}^1.$$

From the equality $[x_{t_2}^2, f] = 0$ we obtain $\alpha_{2,0}^1 = 0$. Thus, we get

$$\left[x_k^2, y_1\right] = \alpha_{2,0}^2 x_k^2, \quad 0 \leqslant k \leqslant t_2.$$

Consider the products

$$\left[\left[x_0^1, h\right], y_1\right] = \left[x_0^1, [h, y_1]\right] + \left[\left[x_0^1, y_1\right], h\right] = \left[\alpha_{1,0}^1 x_0^1 + \alpha_{1,0}^2 x_0^2, h\right] = t_1 \alpha_{1,0}^1 x_0^1 + t_2 \alpha_{1,0}^2 x_0^2.$$

On the other hand

$$\left[\left[x_0^1, h\right], y_1\right] = \left[t_1 x_0^1, y_1\right] = t_1\alpha_{1,0}^1 x_0^1 + t_1\alpha_{1,0}^2 x_0^2.$$

Comparing the coefficient at the basis elements, we have $(t_1 - t_2)\alpha_{1,0}^2 = 0$. The condition $t_1 \neq t_2$, implies $\alpha_{1,0}^2 = 0$.

Thus, we can assume that

$$\left[x_k^1, y_1\right] = \alpha_{1,1} x_k^1, \quad 0 \leqslant k \leqslant t_1,$$
$$\left[x_k^2, y_1\right] = \alpha_{2,1} x_k^2, \quad 0 \leqslant k \leqslant t_2.$$

In a similar way as above we obtain

$$\left[x_k^i, y_j\right] = \alpha_{i,j} x_k^i, \quad 1 \leqslant i \leqslant 2,\ 0 \leqslant k \leqslant t_i,\ 1 \leqslant j \leqslant m.$$

Consider the equalities

$$\begin{aligned}\left[x_k^i, [y_p, y_q]\right] &= \left[\left[x_k^i, y_p\right], y_q\right] - \left[\left[x_k^i, y_q\right], y_p\right] = \left[\alpha_{i,p} x_k^i, y_q\right] - \left[\alpha_{i,q} x_k^i, y_p\right] \\ &= \alpha_{i,p}\alpha_{i,q} x_k^i - \alpha_{i,q}\alpha_{i,p} x_k^i = 0.\end{aligned}$$

Taking into account the property $[S, S] = S$ and arbitrariness of the elements $\{x_k^i, y_p, y_q\}$ we obtain that $[I_1, S] = 0$.

Moreover, $[I_2, sl_2] = 0$. Indeed,

$$[I_2, sl_2] = \left[[I_2, S], sl_2\right] \subseteq \left[I_2, [S, sl_2]\right] + \left[[I_2, sl_2], S\right] = \left[[I_2, sl_2], S\right] \subseteq [I_1, S] = 0.$$

Thus, we have proved that the semisimple Leibniz algebra L is decomposed into the direct sum of two Lie-simple Leibniz algebras, i.e. $L = (sl_2 \dotplus I_1) \oplus (S \dotplus I_2)$. $\square$

Let L be a semisimple Leibniz algebra such that $L = (sl_2^1 \oplus sl_2^2 \oplus \cdots \oplus sl_2^{k-1} \oplus S_k) \dotplus I$ and I_j is a reducible module over sl_2^j. Then the module I_j is completely reducible over sl_2^j, i.e. $I_j = I_{j,1} \oplus I_{j,2} \oplus \cdots \oplus I_{j,p_j}$, where $I_{j,i}$ is irreducible over sl_2^j.

We generalize Proposition 3.5 and define some types of semisimple Leibniz algebras which are decomposed into direct sum of Lie-simple ones.

Theorem 3.6. *If $\dim I_{j,r} \neq \dim I_{j,q}$ for any $1 \leqslant j \leqslant k-1$, $1 \leqslant r, q \leqslant p_j$, $r \neq q$ then*

$$L = \left(sl_2^1 \dotplus I_1\right) \oplus \left(sl_2^2 \dotplus I_2\right) \oplus \cdots \oplus \left(sl_2^{k-1} \dotplus I_{k-1}\right) \oplus (S_k \dotplus I_k).$$

Proof. In order to prove theorem it is sufficient to prove

$$\left[sl_2^j + I_j, S_k + I_k\right] = \left[S_k + I_k, sl_2^j + I_j\right] = 0.$$

Without loss of generality, we can suppose $j = 1$ and $I_1 = I_{1,1} \oplus I_{1,2} \oplus \cdots \oplus I_{1,p}$. Let $\{e, h, f, x_0^1, x_1^1, \ldots, x_{t_1}^1, x_0^2, x_1^2, \ldots, x_{t_2}^2, \ldots x_0^p, x_1^p, \ldots, x_{t_p}^p\}$ be a basis of $sl_2^1 + I_1$ such that

$$\begin{aligned}
\left[x_k^j, h\right] &= (t_j - 2k)x_k^j, && 0 \leqslant k \leqslant t_j, \\
\left[x_k^j, f\right] &= x_{k+1}^j, && 0 \leqslant k \leqslant t_j - 1, \\
\left[x_k^j, e\right] &= -k(t_j + 1 - k)x_{k-1}^j, && 1 \leqslant k \leqslant t_j,
\end{aligned}$$

where $I_{1,j} = \langle x_0^j, x_1^j, \dots, x_{t_j}^j \rangle$, $1 \leqslant j \leqslant p$.

Since $dim\, I_{1,r} \neq dim\, I_{1,q}$, then without loss of generality we can assume that $t_1 > t_2 > \dots > t_p$.

Let $\{y_1, y_2, \dots, y_m\}$ be a basis of S_k. Put

$$\left[x_0^i, y_1\right] = \sum_{j=1}^{p} \sum_{r=0}^{t_j} \alpha_{i,r}^j x_r^j, \quad 1 \leqslant i \leqslant p.$$

Similarly as in the proof of Proposition 3.5 considering Leibniz identity

$$\left[\left[x_k^1, f\right], y_1\right] = \left[x_k^1, [f, y_1]\right] + \left[\left[x_k^1, y_1\right], f\right],$$

we obtain

$$\left[x_k^1, y_1\right] = \sum_{j=1}^{p} \sum_{r=0}^{t_j - k} \alpha_{1,r}^j x_{r+k}^j, \quad 0 \leqslant k \leqslant t_p,$$

$$\left[x_k^1, y_1\right] = \sum_{j=1}^{p-q} \sum_{r=0}^{t_j - k} \alpha_{1,r}^j x_{r+k}^j, \quad 1 \leqslant q \leqslant p - 1,\ t_{p-q+1} + 1 \leqslant k \leqslant t_{p-q}.$$

Consider the equalities

$$\begin{aligned}
\left[\left[x_0^1, e\right], y_1\right] &= \left[x_0^1, [e, y_1]\right] + \left[\left[x_0^1, y_1\right], e\right] = \left[\left[x_0^1, y_1\right], e\right] \\
&= \left[\sum_{j=1}^{p} \sum_{r=0}^{t_j} \alpha_{1,r}^j x_r^j, e\right] = \sum_{j=1}^{p} \sum_{r=1}^{t_j} \alpha_{1,r}^j \bigl(-r(t_j + 1 - r)\bigr) x_{r-1}^j.
\end{aligned}$$

On the other hand

$$\left[\left[x_0^1, e\right], y_1\right] = 0.$$

Comparing the coefficients at the basis elements, we get

$$\alpha_{1,r}^j = 0, \quad 1 \leqslant j \leqslant p,\ 1 \leqslant r \leqslant t_j.$$

Thus, we have

$$\left[x_k^1, y_1\right] = \sum_{j=1}^{p} \alpha_{1,j}^1 x_k^j, \quad 0 \leqslant k \leqslant t_p,$$

$$\left[x_k^1, y_1\right] = \sum_{j=1}^{p-q} \alpha_{1,j}^1 x_k^j, \quad 1 \leqslant q \leqslant p - 1,\ t_{p-q+1} + 1 \leqslant k \leqslant t_{p-1}.$$

Consider the equalities

$$\big[\big[x_0^1, h\big], y_1\big] = \big[x_0^1, [h, y_1]\big] + \big[\big[x_0^1, y_1\big], h\big] = \left[\sum_{j=1}^{p} \alpha_{1,j}^1 x_0^j, h\right] = \sum_{j=1}^{p} t_j \alpha_{1,j}^1 x_0^j.$$

On the other hand

$$\big[\big[x_0^1, h\big], y_1\big] = \big[t_1 x_0^1, y_1\big] = t_1 \sum_{j=1}^{p} \alpha_{1,j}^1 x_0^j.$$

Comparing the coefficient at the basis elements, we have $(t_1 - t_j)\alpha_{1,j}^j = 0$. The condition $t_1 \neq t_j$ implies that $\alpha_{1,j}^j = 0$ for all $2 \leqslant j \leqslant p$.

Thus, rewriting the index of the coefficients, we obtain

$$\big[x_k^1, y_1\big] = \alpha_{1,1} x_k^1, \quad 0 \leqslant k \leqslant t_1.$$

Similarly we obtain

$$\big[x_k^i, y_j\big] = \alpha_{i,j} x_k^i, \quad 1 \leqslant i \leqslant p,\ 0 \leqslant k \leqslant t_i,\ 1 \leqslant j \leqslant m.$$

Consider the products

$$\begin{aligned}\big[x_k^i, [y_p, y_q]\big] &= \big[\big[x_k^i, y_p\big], y_q\big] - \big[\big[x_k^i, y_q\big], y_p\big] = \big[\alpha_{i,p} x_k^i, y_q\big] - \big[\alpha_{i,q} x_k^i, y_p\big] \\ &= \alpha_{i,p}\alpha_{i,q} x_k^i - \alpha_{i,q}\alpha_{i,p} x_k^i = 0.\end{aligned}$$

From the arbitrariness of elements $\{x_k^i, y_p, y_q\}$ and condition $[S_k, S_k] = S_k$ we have that $[I_1, [S_k, S_k]] = [I_1, S_k] = 0$.

From

$$\big[I_k, sl_2^1\big] = \big[[I_k, S_k], sl_2\big] \subseteq \big[I_k, \big[S_k, sl_2^1\big]\big] + \big[\big[I_k, sl_2^1\big], S_k\big] = \big[\big[I_k, sl_2^1\big], S_k\big] \subseteq [I_1, S_k] = 0$$

we get $[I_k, sl_2^1] = 0$.

Thus, we obtain

$$\big[sl_2^j + I_j, S_k + I_k\big] = \big[S_k + I_k, sl_2^j + I_j\big] = 0. \qquad \square$$

Analyzing the proof of Theorem 3.6 we obtain the result, which generalize Example 3.

Theorem 3.7. *Let L be a semisimple Leibniz algebra such that $L = (sl_2 \oplus S) \dot{+} I$ and $I_1 = [I, sl_2]$ is a reducible over sl_2. Let $I_1 = I_{1,1} \oplus I_{1,2} \oplus \cdots \oplus I_{1,p}$, where $I_{1,j}$ is an irreducible over sl_2. If*

$$\dim I_{1,j_1} = \dim I_{1,j_2} = \cdots = \dim I_{1,j_s} = t + 1,$$

then there exist $(t+1)$ pieces of s-dimensional submodules $I_{2,1}, I_{2,2}, \ldots, I_{2,t+1}$ of module $I_2 = [I, S]$ (i.e. $\dim I_{2,i} = s$, $1 \leqslant i \leqslant t+1$) such that

$$I_{2,1} + I_{2,2} + \cdots + I_{2,t+1} = I_1 \cap I_2.$$

Proof. Let L satisfies the condition of theorem. By renumerating of the subindexes $1 := j_1$, $2 := j_2, \dots, s := j_s$, without loss of generality, we can assume that

$$dim\, I_{1,1} = dim\, I_{1,2} = \cdots = dim\, I_{1,s} = t + 1.$$

Analogously as in the proof of Proposition 3.5 considering the Leibniz identity for the products

$$\begin{aligned}
\big[\big[x_k^1, f\big], y_j\big] &= \big[x_k^1, [f, y_j]\big] + \big[\big[x_k^1, y_j\big], f\big], \\
\big[\big[x_k^1, e\big], y_j\big] &= \big[x_k^1, [e, y_j]\big] + \big[\big[x_k^1, y_j\big], e\big], \\
\big[\big[x_k^1, h\big], y_j\big] &= \big[x_k^1, [h, y_j]\big] + \big[\big[x_k^1, y_j\big], h\big],
\end{aligned}$$

we obtain

$$\big[x_k^i, y_j\big] = \sum_{r=1}^{s} \alpha_{j,r}^i x_k^r,$$

where $0 \leqslant k \leqslant t$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant m$.

From these products it is not difficult to see that

$$I_{2,j} = \big\langle x_j^1, x_j^2, \dots, x_j^s \big\rangle, \quad 0 \leqslant j \leqslant t$$

are s-dimensional submodules of I_2 over S. $\square$

Finally, we remark that a semisimple Leibniz algebra is decomposed into a direct sum of Lie-simple ones if and only if $I_p \cap I_q = \{0\}$ for any $p \neq q$ (in denotations of Lemma 3.2).

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