The Derivation Algebra of Gametic Algebra for Linked Loci

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Received 21 October 1986; revised 29 July 1987

ABSTRACT

It is known that a derivation of a genetic algebra has genetic meaning and reflects the symmetries of the algebra. Some well-known results on Lie groups are used to give a structure theorem for the derivation algebra of gametic algebra for linked loci.

1. INTRODUCTION

Consider an infinite randomly mating population, not subject to selection, of diploid individuals which differ in k linked loci. Assume that the number of possible alleles in the mth locus is r_m+1. If U' and U'' are complementary subsets of $K=\{1,2,\ldots,k\}$, we indicate by $U=(U',U'')=(U'',U')$ the partition of K determined by U' and U''. For a given U, we assume that in the zygotes the loci in U' are considered as one block, the loci in U'' as another block, and recombination occurs between the blocks with probability $\lambda(U)$. The gametic inheritance for all k loci has been extensively studied by algebraic methods [1–5]. In this paper, we adopt the approach given by Heuch [2].

To this genetic model there corresponds the real commutative algebra G called the gametic algebra for linked loci. A canonical basis for G may be formally represented by the set of monomials

$$X_{i^*}\equiv X_{1i_1}\cdots X_{ki_k}\qquad (0\leqslant i_m\leqslant r_m,\quad 1\leqslant m\leqslant k),$$

where i^* is the multiindex $(i_1,\ldots,i_k)$, and the multiplication table is given by

$$X_{0^*}^2=X_{0^*}\qquad [0^* \text{ is the multiindex } (0,0,\ldots,0)];$$

$$X_{i^*}X_{j^*}=\begin{cases}\lambda(i^*,j^*)\,X_{i^*+j^*} & \text{if}\quad s(i^*)\cap s(j^*)=\phi,\\ 0 & \text{otherwise.}\end{cases}$$

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$s(i^*)$ is the set $\{t \in K: i_t \neq 0\}$, and $2\lambda(i^*, j^*) = \Sigma\lambda(U)$, where the sum is taken over the collection of all partitions U with $s(i^*)$ contained in one of the sets U' or U'' and $s(j^*)$ in the other.

A derivation d of G is a linear map $d: G \to G$ verifying $d(xy) = d(x)y + xd(y)$. The set $\mathrm{Der}(G)$ of all derivations is closed under the Lie bracket $[d, d'] = dd' - d'd$, and it is a Lie algebra.

Holgate [7] gave an explanation of the genetic meaning of derivations in genetic algebras. Briefly, if we have a genetically determined trait with array of values d and d is a derivation, then the above equation verified by d reflects a kind of symmetry in the time direction.

The purpose of this paper is to determine the derivations of G. This is done by using Lie group theory.

2. AUTOMORPHISMS

An automorphism of G is a nonsingular linear map $\psi: G \to G$ that preserves products: $\psi(xy) = \psi(x)\psi(y)$. The collection of all automorphisms of G is a group, which we indicate by $\mathrm{Aut}(G)$.

In this section, we shall get some results about automorphisms that we shall use to obtain the derivations of G.

Given a k-tuple $a = (a_1, \ldots, a_k)$, where a_m $(1 \leqslant m \leqslant k)$ is an $(r_m + 1) \times (r_m + 1)$ real matrix (a_{mij}), and multiindices i^*, j^*, we denote the product $a_{1i_1j_1} \cdots a_{ki_kj_k}$ by $a_{i^*j^*}$. We define the linear map $\bar{a}$ of G by setting

$$\bar{a}(X_{j^*}) = \sum_{i^* = 0^*}^{r^*} a_{i^*j^*} X_{i^*}.$$

Let $A(m)$ $(1 \leqslant m \leqslant k)$ be the affine group of $\mathbf{R}^{r_m}$ (i.e., the set of all nonsingular matrices a_m with $a_{m00} = 1$ and $a_{m0j} = 0$, $1 \leqslant j \leqslant r_m$). We designate by $\bar{A}$ the set of all $\bar{a}$ with a in the direct product $A(1) \times \cdots \times A(k)$.

PROPOSITION

(i) *The map* $a \in A(1) \times \cdots \times A(k) \to \bar{a} \in \bar{A}$ *is an isomorphism of groups.*
(ii) $\bar{A}$ *is a subgroup of* $\mathrm{Aut}(G)$.

Proof. Statement (i) is clear. In order to prove (ii) it is enough to show that $\bar{a}$ preserves products, since $\bar{a}$ is nonsingular by (i). From [3, p. 37] it follows that $\bar{a}(X_{0^*})$ is an idempotent of G and thus $\bar{a}$ preserves the product $X_{0^*}X_{0^*}$. Let p^*, q^* be multiindices with $q^* \neq 0^*$. To prove that $\bar{a}$ preserves $X_{p^*}X_{q^*}$, we shall analyze two cases: $s(p^*) \cap s(q^*) \neq \phi$ and $s(p^*) \cap s(q^*) = \phi$. We have

$$\bar{a}(X_{p^*})\bar{a}(X_{q^*}) = \sum_{m^*, n^* = 0^*}^{r^*} a_{m^*p^*}a_{n^*p^*}X_{m^*}X_{n^*}.$$

Since $a_{i0p_i} = a_{j0q_j} = 0$ for any $i \in s(p^*)$ and $j \in s(q^*)$, we assume that in the above sum m^*, n^* are such that $s(p^*) \subset s(m^*)$ and $s(q^*) \subset s(n^*)$. When $s(p^*) \cap s(q^*) \neq \phi$, we have $s(m^*) \cap s(n^*) \neq \phi$ and thus $\bar{a}(X_{p^*})\bar{a}(X_{q^*}) = 0 = \bar{a}(X_{p^*}X_{q^*})$. Now, suppose that $s(p^*) \cap s(q^*) = \phi$. The component of $\bar{a}(X_{p^*})\bar{a}(X_{q^*})$ in the direction of X_{t^*}, in the case where $s(p^*) \cup s(q^*) \subset s(t^*)$, is $[\Sigma\lambda(m^*, n^*)]a_{t^*, p^*+q^*}$; the sum is taken over all m^*, n^* with $m^* + n^* = t^*$, $s(m^*) \cap s(n^*) = \phi$, $s(p^*) \subset s(m^*)$, and $s(q^*) \subset s(n^*)$. As is readily seen, $\Sigma\lambda(m^*, n^*) = \lambda(p^*, q^*)$, and so the component is equal to $\lambda(p^*, q^*)a_{t^*, p^*+q^*}$. When $s(p^*) \cup s(q^*) \not\subset s(t^*)$, this component is zero. From this, it is now clear that

$$\bar{a}(X_{p^*})\bar{a}(X_{q^*}) = \bar{a}(X_{p^*}X_{q^*}).$$

We notice that if $r_i \neq r_j$ for all $i, j \in K$, $i \neq j$, we have $\mathrm{Aut}(G) = \bar{A}$ and then $\mathrm{Aut}(G)$ is isomorphic to $A(1) \times \cdots \times A(k)$. When $r_i = r_j$ for some $i \neq j$, it seems that whether $\bar{A}$ is or is not the automorphism group of G depends on the linkage distribution $\{\lambda(U)\}$. We do not give details here, because these facts are not necessary to prove the main result of the paper.

3. DERIVATIONS

The affine group of $\mathbf{R}^{r_m}$ is a Lie group, and its Lie algebra is the vector space $L(m) \equiv \mathbf{R}^{r_m} \oplus \mathrm{gl}(r_m, \mathbf{R})$ with the Lie bracket $[h, h'] = (c\delta' - c'\delta, cc' - c'c)$, where $\mathrm{gl}(r_m, \mathbf{R})$ denotes the set of all $r_m \times r_m$ real matrices, $h = (\delta, c)$, and $h' = (\delta', c')$ [11, p. 192]. On the other hand, it is well known that if $G_1, \ldots, G_k$ are Lie groups with Lie algebras $L_1, \ldots, L_k$, then the Lie algebra of the direct product $G_1 \times \cdots \times G_k$ is the direct product of Lie algebras $L_1 \times \cdots \times L_k$ [9, p. 10-9]. Therefore, the Lie algebra of $A(1) \times \cdots \times A(k)$ is $L(1) \times \cdots \times L(k)$.

From the proposition, we have that the map

$$a \in A(1) \times \cdots \times A(k) \xrightarrow{\theta} \bar{a} \in \mathrm{Aut}(G)$$

is a homomorphism of (abstract) groups; furthermore, since the coordinate functions of θ are polynomials in the indeterminates a_{mij}, θ is C^∞. We may thus conclude that θ is a homomorphism of Lie groups. Therefore, the differential map

$$d\theta : L(1) \times \cdots \times L(k) \to \mathrm{Der}(G)$$

is a homomorphism of Lie algebras (see [10, Theorem 3.14] and recall that the Lie algebra of the automorphism group of G is its derivation algebra $\mathrm{Der}(G)$ [10, Theorem 3.54]). Hence, given a k-tuple $h = (h_1, \ldots, h_k)$, where $(h_{mij}) \in L(m)$, $d = d\theta \cdot h$ is a derivation of G.

The coordinate function of θ in the direction of (X_{i^*}, X_{j^*}) [i.e., the function that maps the element a of $A(1)\times\cdots\times A(k)$ to the coordinate of $\bar{a}(X_{i^*})$ in the direction of X_{j^*}] is

$$a \xrightarrow{\psi} a_{1j_1i_1}\cdots a_{kj_ki_k}.$$

Since this is a polynomial function, it is easy to obtain its partial derivatives, and then to conclude that

$$d\psi\cdot h = h_{mt_mi_m} \quad \text{if } X_{1j_1}\cdots X_{kj_k} = X_{mt_m}X_{1i_1}\cdots \hat{X}_{mi_m}\cdots X_{ki_k} \qquad (1\leqslant t_m\leqslant r_m)$$

($\hat{}$ denotes absence) and $d\psi\cdot h = 0$ in any other case. Therefore,

$$d(X_{i^*}) = \sum_{m=1}^{k}\sum_{t_m=1}^{r_m} h_{mt_mi_m}X_{mt_m}\hat{X}_{i^*}, \tag{1}$$

where $X_{mt_m}\hat{X}_{i^*}$ denotes $X_{mt_m}X_{i_1}\cdots\hat{X}_{mi_m}\cdots X_{ki_k}$.

It is easy to see that $d\theta$ is injective. Now, we claim that $d\theta$ is onto. Let d be a derivation of G. We must prove that d is given by (1). Write

$$d(X_{i^*}) = \sum_{t^*=0^*}^{r^*}\alpha_{t^*i^*}X_{t^*} \qquad (\alpha_{t^*i^*}\in\mathbf{R}).$$

If $\omega: G\to\mathbf{R}$ is the linear form defined by $\omega(X_{0^*})=1$, $\omega(X_{t^*})=0$ $(t^*\neq 0^*)$, then $\omega d = 0$ [6, Theorem 1]. It follows thus that $a_{0^*i^*}=0$. Since X_{0^*} is an idempotent, we have $d(X_{0^*}) = 2X_{0^*}d(X_{0^*})$ and so

$$\sum_{t^*=0^*}^{r^*}[1-2\lambda(0^*,t^*)]\alpha_{t^*0^*}X_{t^*}=0.$$

Thus, if t^* has at least two nonzero coordinates, we have that $a_{t^*}=0$, since $2\lambda(0^*,t^*)<1$ (as noticed in [3, p. 37], we may assume this condition without loss of generality). Next, assume that the only nonzero coordinate of i^* is the sth. Since d is derivation, we obtain

$$\frac{1}{2}\sum_{t^*=0^*}^{r^*}\alpha_{t^*i^*}X_{t^*} = X_{0^*}\left(\sum_{t^*=0^*}^{r^*}\alpha_{t^*i^*}X_{t^*}\right) + X_{i^*}\left(\sum_{m=1}^{k}\sum_{t_m=1}^{r_m}h_{mt_m0}X_{mt_m}\hat{X}_{0^*}\right).$$

Hence, if t^* has more than two nonzero coordinates or if t^* has exactly two nonzero coordinates and neither of them is the sth, we have (by the same argument used above) that $\alpha_{t^*i^*} = 0$. On the other hand, let $X_{p^*} = X_{mt_m}\hat{X}_{0^*}$ $(m \neq s)$. If $t^* = p^* + i^*$ then

$$\left[\tfrac{1}{2} - \lambda(0^*, t^*)\right] \alpha_{t^*i^*} = \lambda(p^*, i^*) h_{mt_m0}$$

and so $\alpha_{t^*i^*} = h_{mt_m0}$. Also, from $X_{i^*} d(X_{i^*}) = 0$ we get $\alpha_{t^*i^*} = 0$, when t^* has one nonzero coordinate and it is not the sth. Therefore, $d(X_{i^*})$ is given by (1). Now, we finish the proof of our claim by induction. Assume that $d(X_{i^*})$ has the form (1) for any i^* with no more than $t-1$ nonzero coordinates. Let i^* be a multiindex with t nonzero coordinates, and take nonzero p^*, q^* such that $p^* + q^* = i^*$ and $s(p^*) \cap s(q^*) = \phi$. Let t_m be such that $1 \leqslant t_m \leqslant r_m$. For $m \in K$ with $i_m \neq 0$ we obtain

$$X_{q^*}\left(X_{mt_m}\hat{X}_{p^*}\right) = \begin{cases} \lambda(p^*, q^*) X_{mt_m}\hat{X}_{i^*} & \text{if} \quad q_m = 0, \\ 0 & \text{if} \quad q_m \neq 0, \end{cases}$$

and

$$X_{p^*}\left(X_{mt_m}\hat{X}_{q^*}\right) = \begin{cases} \lambda(p^*, q^*) X_{mt_m}\hat{X}_{i^*} & \text{if} \quad p_m = 0, \\ 0 & \text{if} \quad p_m \neq 0. \end{cases}$$

On the other hand, for $m \in K$ with $i_m = 0$, we get

$$X_{q^*}\left(X_{mt_m}\hat{X}_{p^*}\right) = \lambda(q^*, n^*) X_{mt_m}\hat{X}_{i^*}$$

and

$$X_{p^*}\left(X_{mt_m}\hat{X}_{q^*}\right) = \lambda(p^*, m^*) X_{mt_m}\hat{X}_{i^*},$$

where m^* (n^*) is the sum of q^* (p^*, respectively) with the multiindex whose sole nonzero coordinate is the mth and is equal to t_m. It is immediate to verify that $\lambda(p^*, m^*) + \lambda(q^*, n^*) = \lambda(p^*, q^*)$. Thus, using again the fact that d is a derivation, we have

$$\begin{aligned} \lambda(p^*, q^*) d(X_{i^*}) &= X_{q^*}\left(\sum_{m=1}^{k} \sum_{t_m=1}^{r_m} h_{mt_mp_m} X_{mt_m}\hat{X}_{p^*}\right) \\ &\quad + X_{p^*}\left(\sum_{m=1}^{k} \sum_{t_m=1}^{r_m} h_{mt_mq_m} X_{mt_m}\hat{X}_{q^*}\right) \\ &= \lambda(p^*, q^*) \sum_{m=1}^{k} \sum_{t_m=1}^{r_m} h_{mt_mi_m} X_{mt_m}\hat{X}_{i^*}. \end{aligned}$$

Therefore, $d(X_{i^*})$ is given by (1).

These considerations can be summarized as follows:

THEOREM

The derivation algebra of G is isomorphic to the direct product of Lie algebras $L(1)\times \cdots \times L(k)$.

The derivations of the corresponding zygotic algebra can be easily obtained from (1); see [8].

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