The Derivation Algebra of Gametic Algebra for Linked Loci

LUIZ A. PERESI

Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 20570, São Paulo, Brazil 01498

Received 21 October 1986; revised 29 July 1987

ABSTRACT

It is known that a derivation of a genetic algebra has genetic meaning and reflects the symmetries of the algebra. Some well-known results on Lie groups are used to give a structure theorem for the derivation algebra of gametic algebra for linked loci.

1. INTRODUCTION

Consider an infinite randomly mating population, not subject to selection, of diploid individuals which differ in $k$ linked loci. Assume that the number of possible alleles in the $m$th locus is $r_m + 1$. If $U'$ and $U''$ are complementary subsets of $K = \{1, 2, \ldots, k\}$, we indicate by $U = (U', U'') = (U', U')$ the partition of $K$ determined by $U'$ and $U''$. For a given $U$, we assume that in the zygotens the loci in $U'$ are considered as one block, the loci in $U''$ as another block, and recombination occurs between the blocks with probability $\lambda(U)$. The gametic inheritance for all $k$ loci has been extensively studied by algebraic methods [1-5]. In this paper, we adopt the approach given by Heuch [2].

To this genetic model there corresponds the real commutative algebra $G$ called the gametic algebra for linked loci. A canonical basis for $G$ may be formally represented by the set of monomials

$$X_{i^*} = X_{i_1} \cdots X_{i_k} \quad (0 \leq i_m \leq r_m, \quad 1 \leq m \leq k),$$

where $i^*$ is the multiindex $(i_1, \ldots, i_k)$, and the multiplication table is given by

$$X_{0^*}^2 = X_{0^*}, \quad [0^* \text{ is the multiindex } (0,0,\ldots,0)];$$

$$X_{i^*} X_{j^*} = \begin{cases} \lambda(i^*, j^*) X_{i^*+j^*} & \text{if } s(i^*) \cap s(j^*) = \phi, \\ 0 & \text{otherwise.} \end{cases}$$
s(i*) is the set \( \{ t \in K : i, \neq 0 \} \), and \( 2\lambda(i^*, j^*) = \sum \lambda(U) \), where the sum is taken over the collection of all partitions \( U \) with \( s(i^*) \) contained in one of the sets \( U' \) or \( U'' \) and \( s(j^*) \) in the other.

A derivation \( d \) of \( G \) is a linear map \( d : G \rightarrow G \) verifying \( d(xy) = d(x)y + xd(y) \). The set \( \text{Der}(G) \) of all derivations is closed under the Lie bracket \( [d, d'] = dd' - d'd \), and it is a Lie algebra.

Holgate [7] gave an explanation of the genetic meaning of derivations in genetic algebras. Briefly, if we have a genetically determined trait with array of values \( d \) and \( d \) is a derivation, then the above equation verified by \( d \) reflects a kind of symmetry in the time direction.

The purpose of this paper is to determine the derivations of \( G \). This is done by using Lie group theory.

2. AUTOMORPHISMS

An automorphism of \( G \) is a nonsingular linear map \( \psi : G \rightarrow G \) that preserves products: \( \psi(xy) = \psi(x)\psi(y) \). The collection of all automorphisms of \( G \) is a group, which we indicate by \( \text{Aut}(G) \).

In this section, we shall get some results about automorphisms that we shall use to obtain the derivations of \( G \).

Given a \( k \)-tuple \( a = (a_1, \ldots, a_k) \), where \( a_m \) (\( 1 \leq m \leq k \)) is an \((r_m + 1) \times (r_m + 1)\) real matrix \((a_{mij})\), and multiindices \( i^*, j^* \), we denote the product \( a_{1i_1j_1} \cdots a_{ki_kj_k} \) by \( a_{i^*, j^*} \). We define the linear map \( \bar{a} \) of \( G \) by setting

\[
\bar{a}(X_{i^*}) = \sum_{i^* = 0^*} a_{i^*, j^*} X_{j^*}.
\]

Let \( A(m) \) (\( 1 \leq m \leq k \)) be the affine group of \( \mathbb{R}^{rm} \) (i.e., the set of all nonsingular matrices \( a_m \) with \( a_{m00} = 1 \) and \( a_{m0j} = 0 \), \( 1 \leq j \leq r_m \)). We designate by \( \overline{A} \) the set of all \( \bar{a} \) with \( a \) in the direct product \( A(1) \times \cdots \times A(k) \).

**PROPOSITION**

(i) The map \( a \in A(1) \times \cdots \times A(k) \rightarrow \bar{a} \in \overline{A} \) is an isomorphism of groups.

(ii) \( \overline{A} \) is a subgroup of \( \text{Aut}(G) \).

**Proof.** Statement (i) is clear. In order to prove (ii) it is enough to show that \( \bar{a} \) preserves products, since \( \bar{a} \) is nonsingular by (i). From [3, p. 371] it follows that \( \bar{a}(X_{0^*}) \) is an idempotent of \( G \) and thus \( \bar{a} \) preserves the product \( X_{0^*}X_{0^*} \). Let \( p^*, q^* \) be multiindices with \( q^* \neq 0^* \). To prove that \( \bar{a} \) preserves \( X_{p^*}X_{q^*} \), we shall analyze two cases: \( s(p^*) \cap s(q^*) \neq \emptyset \) and \( s(p^*) \cap s(q^*) = \emptyset \). We have

\[
\bar{a}(X_{p^*}) \bar{a}(X_{q^*}) = \sum_{m^*, n^* = 0^*} a_{m^*p^*} a_{n^*q^*} X_{m^*}X_{n^*}.
\]
DERIVATIONS FOR LINKED LOCI

Since \( a_{i0p} = a_{j0q} = 0 \) for any \( i \in s(p^*) \) and \( j \in s(q^*) \), we assume that in the above sum \( m^*, n^* \) are such that \( s(p^*) \subseteq s(m^*) \) and \( s(q^*) \subseteq s(n^*) \). When \( s(p^*) \cap s(q^*) \neq \emptyset \), we have \( s(m^*) \cap s(n^*) \neq \emptyset \) and thus \( \bar{a}(X_{p^*})\bar{a}(X_{q^*}) = 0 = \bar{a}(X_{p^*}X_{q^*}) \). Now, suppose that \( s(p^*) \cap s(q^*) = \emptyset \). The component of \( \bar{a}(X_{p^*})\bar{a}(X_{q^*}) \) in the direction of \( X_{p^*} \), in the case where \( s(p^*) \cup s(q^*) \subseteq s(t^*) \), is \( [\sum\lambda(m^*, n^*)]a_{r^*, r^* + t^*}; \) the sum is taken over all \( m^*, n^* \) with \( m^* + n^* = t^* \), \( s(m^*) \cap s(n^*) = \emptyset \), and \( s(p^*) \subseteq s(m^*) \), and \( s(q^*) \subseteq s(n^*) \). As is readily seen, \( \sum\lambda(m^*, n^*) = \lambda(p^*, q^*) \), and so the component is equal to \( \lambda(p^*, q^*)a_{r^*, r^* + q^*} \). When \( s(p^*) \cup s(q^*) \not\subseteq s(t^*) \), this component is zero. From this, it is now clear that

\[
\bar{a}(X_{p^*})\bar{a}(X_{q^*}) = \bar{a}(X_{p^*}X_{q^*}).
\]

We notice that if \( r_i \neq r_j \) for all \( i, j \in K, i \neq j \), we have \( \text{Aut}(G) = \bar{A} \) and then \( \text{Aut}(G) \) is isomorphic to \( A(1) \times \cdots \times A(k) \). When \( r_i = r_j \) for some \( i = j \), it seems that whether \( \bar{A} \) is or is not the automorphism group of \( G \) depends on the linkage distribution \( \{\lambda(U)\} \). We do not give details here, because these facts are not necessary to prove the main result of the paper.

3. DERIVATIONS

The affine group of \( \mathbb{R}^m \) is a Lie group, and its Lie algebra is the vector space \( L(m) = \mathbb{R}^m \oplus gl(r, \mathbb{R}) \) with the Lie bracket \([h, h'] = (c\delta' - c'\delta, cc' - c'c)\), where \( gl(r, \mathbb{R}) \) denotes the set of all \( r \times r \) real matrices, \( h = (\delta, c) \), and \( h' = (\delta', c') \) [11, p. 192]. On the other hand, it is well known that if \( G_1, \ldots, G_k \) are Lie groups with Lie algebras \( L_1, \ldots, L_k \), then the Lie algebra of the direct product \( G_1 \times \cdots \times G_k \) is the direct product of Lie algebras \( L_1 \times \cdots \times L_k \) [9, p. 10-9]. Therefore, the Lie algebra of \( A(1) \times \cdots \times A(k) \) is \( L(1) \times \cdots \times L(k) \).

From the proposition, we have that the map

\[
a \in A(1) \times \cdots \times A(k) \rightarrow \bar{a} \in \text{Aut}(G)
\]

is a homomorphism of (abstract) groups; furthermore, since the coordinate functions of \( \theta \) are polynomials in the indeterminates \( a_{mij} \), \( \theta \) is \( C^\infty \). We may thus conclude that \( \theta \) is a homomorphism of Lie groups. Therefore, the differential map

\[
d\theta : L(1) \times \cdots \times L(k) \rightarrow \text{Der}(G)
\]

is a homomorphism of Lie algebras (see [10, Theorem 3.14] and recall that the Lie algebra of the automorphism group of \( G \) is its derivation algebra \( \text{Der}(G) \) [10, Theorem 3.54]). Hence, given a \( k \)-tuple \( h = (h_1, \ldots, h_k) \), where \( (h_{mij}) \in L(m) \), \( d = d\theta \cdot h \) is a derivation of \( G \).
The coordinate function of \( \theta \) in the direction of \((X_{*,1}, \ldots, X_{*,s})\) [i.e., the function that maps the element \( a \) of \( A(1) \times \cdots \times A(k) \) to the coordinate of \( \alpha(X_{*,s}) \) in the direction of \( X_{*,s} \)] is

\[
a \mapsto a_{i_1j_1} \cdots a_{i_kj_k}.
\]

Since this is a polynomial function, it is easy to obtain its partial derivatives, and then to conclude that

\[
d\psi \cdot h = h_{m_{1\ldots m}} \quad \text{if} \quad X_{1_{i_1}} \cdots X_{k_{i_k}} = X_{m_{1\ldots m}} X_{k_{i_1}} \cdots \hat{X}_{m_{1\ldots m}} \cdots X_{k_{i_k}} \quad (1 \leq m \leq r_m)
\]

(\( \hat{\cdot} \) denotes absence) and \( d\psi \cdot h = 0 \) in any other case. Therefore,

\[
d(X_{*,s}) = \sum_{m=1}^k \sum_{n=1}^{r_m} h_{m_{1\ldots m}} X_{m_{1\ldots m}} \hat{X}_{i_{*,s}},
\]

where \( X_{m_{1\ldots m}} \hat{X}_{i_{*,s}} \) denotes \( X_{m_{1\ldots m}} X_{i_{1\ldots i_{*,s}}} \cdots \hat{X}_{m_{1\ldots m}} \cdots X_{k_{i_k}} \).

It is easy to see that \( d\theta \) is injective. Now, we claim that \( d\theta \) is onto. Let \( d \) be a derivation of \( G \). We must prove that \( d \) is given by (1). Write

\[
d(X_{*,s}) = \sum_{t^* = 0^*}^{r^*} \alpha_{s_{*,t^*}} X_{s_{*,t^*}} \quad (\alpha_{s_{*,t^*}} \in \mathbb{R}).
\]

If \( \omega : G \to \mathbb{R} \) is the linear form defined by \( \omega(X_{0_{*,t^*}}) = 1, \omega(X_{*,0^*}) = 0 \) \((t^* \neq 0^*)\), then \( \omega d = 0 \) \([6, \text{Theorem 1}]\). It follows thus that \( a_{0_{*,t^*}} = 0 \). Since \( X_{0_{*,t^*}} \) is an idempotent, we have \( d(X_{0_{*,t^*}}) = 2 X_{0_{*,t^*}} d(X_{0_{*,t^*}}) \) and so

\[
\sum_{t^* = 0^*}^{r^*} [1 - 2\lambda(0^*, t^*)] \alpha_{s_{*,t^*}} X_{s_{*,t^*}} = 0.
\]

Thus, if \( t^* \) has at least two nonzero coordinates, we have that \( a_{s_{*,t^*}} = 0 \), since \( 2\lambda(0^*, t^*) < 1 \) (as noticed in \([3, \text{p. 37}]\), we may assume this condition without loss of generality). Next, assume that the only nonzero coordinate of \( t^* \) is the \( s \)th. Since \( d \) is derivation, we obtain

\[
\frac{1}{2} \sum_{t^* = 0^*}^{r^*} \alpha_{s_{*,t^*}} X_{s_{*,t^*}} = X_{0_{*,t^*}} \left( \sum_{t^* = 0^*}^{r^*} \alpha_{s_{*,t^*}} X_{s_{*,t^*}} \right)
\]

\[
+ X_{s_{*,t^*}} \left( \sum_{m=1}^k \sum_{n=1}^{r_m} h_{m_{1\ldots m}} X_{m_{1\ldots m}} \hat{X}_{0_{*,t^*}} \right).
\]
Hence, if \( t^* \) has more than two nonzero coordinates or if \( t^* \) has exactly two nonzero coordinates and neither of them is the \( s \)th, we have (by the same argument used above) that \( \alpha_{r^*,s} = 0 \). On the other hand, let \( X_{p^*} = X_{mt_m} \hat{X}_0 \). If \( t^* = p^* + i^* \) then

\[
\left[ \frac{1}{2} - \lambda(0^*, t^*) \right] \alpha_{r^*,s} = \lambda(p^*, i^*) h_{mt_m,0}
\]

and so \( \alpha_{r^*,s} = h_{mt_m,0} \). Also, from \( X_{i^*} d(X_{i^*}) = 0 \) we get \( \alpha_{r^*,s} = 0 \), when \( t^* \) has one nonzero coordinate and it is not the \( s \)th. Therefore, \( d(X_{i^*}) \) is given by (1). Now, we finish the proof of our claim by induction. Assume that \( d(X_{i^*}) \) has the form (1) for any \( i^* \) with no more than \( t - 1 \) nonzero coordinates. Let \( i^* \) be a multiindex with \( t \) nonzero coordinates, and take nonzero \( p^*, q^* \) such that \( p^* + q^* = i^* \) and \( s(p^*) \cap s(q^*) = \emptyset \). Let \( t_m \) be such that \( 1 \leq t_m \leq r_m \). For \( m \in K \) with \( i_m \neq 0 \) we obtain

\[
X_{q^*}(X_{mt_m} \hat{X}_{p^*}) = \begin{cases} 
\lambda(p^*, q^*) X_{mt_m} \hat{X}_{i^*} & \text{if } q_m = 0, \\
0 & \text{if } q_m \neq 0,
\end{cases}
\]

and

\[
X_{p^*}(X_{mt_m} \hat{X}_{q^*}) = \begin{cases} 
\lambda(p^*, q^*) X_{mt_m} \hat{X}_{i^*} & \text{if } p_m = 0, \\
0 & \text{if } p_m \neq 0.
\end{cases}
\]

On the other hand, for \( m \in K \) with \( i_m = 0 \), we get

\[
X_{q^*}(X_{mt_m} \hat{X}_{p^*}) = \lambda(q^*, n^*) X_{mt_m} \hat{X}_{i^*}
\]

and

\[
X_{p^*}(X_{mt_m} \hat{X}_{q^*}) = \lambda(p^*, m^*) X_{mt_m} \hat{X}_{i^*},
\]

where \( m^* (n^*) \) is the sum of \( q^* (p^* \), respectively) with the multiindex whose sole nonzero coordinate is the \( m \)th and is equal to \( t_m \). It is immediate to verify that \( \lambda(p^*, m^*) + \lambda(q^*, n^*) = \lambda(p^*, q^*) \). Thus, using again the fact that \( d \) is a derivation, we have

\[
\lambda(p^*, q^*) d(X_{i^*}) = X_{q^*} \left( \sum_{m=1}^{k} \sum_{i_m=1}^{r_m} h_{mt_m,p^*} X_{mt_m} \hat{X}_{p^*} \right)
\]

\[
+ X_{p^*} \left( \sum_{m=1}^{k} \sum_{i_m=1}^{r_m} h_{mt_m,q^*} X_{mt_m} \hat{X}_{q^*} \right)
\]

\[
= \lambda(p^*, q^*) \sum_{m=1}^{k} \sum_{i_m=1}^{r_m} h_{mt_m,i^*} X_{mt_m} \hat{X}_{i^*}.
\]

Therefore, \( d(X_{i^*}) \) is given by (1).
These considerations can be summarized as follows:

**THEOREM**

The derivation algebra of $G$ is isomorphic to the direct product of Lie algebras $L(1) \times \cdots \times L(k)$.

The derivations of the corresponding zygotic algebra can be easily obtained from (1); see [8].

**REFERENCES**