A Note on Duplication of Algebras

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ABSTRACT

It is proved that if A is a nonassociative algebra that verifies $A^2 = A$ and has an idempotent, then A and its duplicate have isomorphic automorphism groups and isomorphic derivation algebras. The result is then applied to the gametic algebra for polyploidy with multiple alleles.

1. PRELIMINARIES

The concept of duplicate of a nonassociative algebra was introduced by Etherington, [1, 2] and it has been used in the study of zygotic populations by algebraic methods (see [4]).

Let K be a commutative ring with a unit element, and A a K-algebra. The duplicate of A consists of the tensor product K-module $A \otimes A$ with the multiplication

$$(a \otimes b)(c \otimes d) = ab \otimes cd.$$

We shall indicate the duplicate of A by $A \otimes A$ as well. Now, suppose that A is a commutative algebra. The duplicate of A is in general noncommutative. But the quotient algebra $A \otimes A / I$, where I is the ideal of A generated as a submodule by the elements $a \otimes b - b \otimes a$ $(a, b \in A)$, is commutative. $A \otimes A / I$ is called the commutative duplicate of A.

The map $\sum_i a_i \otimes b_i \in A \otimes A \to \sum_i a_i b_i \in A^2$ is K-linear and surjective, and its kernel N is an ideal of annihilators, i.e., for any $x \in A \otimes A$ and $t \in N$, we

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have $xt = tx = 0$. Then, in the case where $A^2 = A$, we have that A is isomorphic to $A\otimes A/N$; under this isomorphism, to the element $x = \sum_i x_i' x_i''$ of A there corresponds the element $\sum_i x_i' \otimes x_i'' + N$, and we shall indicate this fact by writing $x \equiv \sum_i x_i' \otimes x_i'' + N$. An analogous result can be obtained for the commutative duplicate of a commutative algebra A such that $A^2 = A$.

2. AUTOMORPHISMS. DERIVATIONS

Let d be a derivation and ψ be an automorphism of A. The maps

$$(x, y) \in A \times A \to d(x) \otimes y + x \otimes d(y) \in A \otimes A$$

and

$$(x, y) \in A \times A \to \psi(x) \otimes \psi(y) \in A \otimes A$$

are K-bilinear. Thus, the universal property of tensor product yields K-linear operators $d_\otimes$ and $\psi^\otimes$ of $A\otimes A$ such that

$$d_\otimes(x \otimes y) = d(x) \otimes y + x \otimes d(y), \quad \psi^\otimes(x \otimes y) = \psi(x) \otimes \psi(y) \qquad (x, y \in A).$$

An easy calculation shows that $d_\otimes$ and $\psi^\otimes$ are, respectively, a derivation and an automorphism of $A\otimes A$, and that the maps $d \to d_\otimes$ and $\psi \to \psi^\otimes$ are, respectively, a homomorphism of Lie algebras and a homomorphism of groups. The fact that $\psi^\otimes$ is an automorphism was pointed out by Etherington [2, Theorem 4].

Proposition. *Assume that* $A^2 = A$. *Then, the map*

$$d \in \mathrm{Der}(A) \to d_\otimes \in \mathrm{Der}(A \otimes A) \qquad [\psi \in \mathrm{Aut}(A) \to \psi^\otimes \in \mathrm{Aut}(A \otimes A)]$$

is an isomorphism if and only if $\bar{d}(N) \subset N$ $[\bar{\psi}(N) \subset N]$ *for any derivation* $\bar{d}$ [*any automorphism* $\bar{\psi}$, *respectively*] *of* $A\otimes A$.

Proof. As is clear, $\bar{d}(N) \subset N$ for any derivation $\bar{d}$ of $A\otimes A$ is a necessary condition for $d \to d_\otimes$ be an isomorphism. On the other hand, let $\bar{d}$ be a derivation of $A\otimes A$, and assume that $\bar{d}(N) \subset N$. Then, the K-linear operator d: $A \to A$ defined by $d(x) \equiv \bar{d}(\sum_i x_i' \otimes x_i'') + N$ is well defined and it is a derivation, since

$$d(x)y + xd(y) \equiv \bar{d}\left[\left(\sum_i x_i' \otimes x_i''\right)\left(\sum_j y_j' \otimes y_j''\right)\right] + N = \bar{d}(x \otimes y) + N \equiv d(xy)$$

for any x, y in A. We claim that $\bar{d} = d_{\otimes}$. For any x, y in A, we have

$$\begin{aligned}\bar{d}(x \otimes y) + N &\equiv d(x)y + xd(y) \\ &\equiv [d(x) \otimes y + x \otimes d(y)] + N = d_{\otimes}(x \otimes y) + N,\end{aligned}$$

and so $\bar{d}(x \otimes y) = d_{\otimes}(x \otimes y) + n$ for some $n \in N$. But then writing $\bar{d}(x_i' \otimes x_i'') = d_{\otimes}(x_i' \otimes x_i'') + n_i$ and $\bar{d}(y_j' \otimes y_j'') = d_{\otimes}(y_j' \otimes y_j'') + m_j$ ($n_i, m_j \in N$), we get

$$\begin{aligned}\bar{d}(x \otimes y) &= \sum_{i,j} \bar{d}\left[(x_i' \otimes x_i'')(y_j' \otimes y_j'')\right] \\ &= \sum_{i,j} \left\{ \left[d_{\otimes}(x_i' \otimes x_i'') + n_i\right](y_j' \otimes y_j'') + (x_i' \otimes x_i'')\left[d_{\otimes}(y_j' \otimes y_j'') + m_j\right] \right\} \\ &= d_{\otimes}(x \otimes y),\end{aligned}$$

since N is an ideal of annihilators. The assertion is proved. This shows that, if $\bar{d}(N) \subset N$ for any derivation $\bar{d}$ of $A \otimes A$, then the map $d \to d_{\otimes}$ is surjective. Now, let d be a derivation of A, and suppose that $d_{\otimes} = 0$. Then, for any $x \in A$, we have

$$\begin{aligned}d(x) &= \sum_i \left\{ d(x_i')x_i'' + x_i'd(x_i'') \right\} \equiv \sum_i \left\{ d(x_i') \otimes x_i'' + x_i' \otimes d(x_i'') \right\} + N \\ &= \sum_i d_{\otimes}(x_i' \otimes x_i'') + N \equiv 0,\end{aligned}$$

i.e., $d = 0$. Under that condition, the map $d \to d_{\otimes}$ is thus an isomorphism.
The proof of the second part of the proposition is similar. ∎

COROLLARY. *Let A be a nonassociative algebra over a field F. Assume that $A^2 = A$ and that A has an idempotent e. Then, the maps $d \in$ Der$(A) \to d_{\otimes} \in$ Der$(A \otimes A)$ and $\psi \in$ Aut$(A) \to \psi^{\otimes} \in$ Aut$(A \otimes A)$ are isomorphisms.*

Proof. We shall prove that $\bar{d}(n) \in N$, where $\bar{d}$ is a derivation of $A \otimes A$ and $n \in N$. Let $\{e, e_i\}_{i \in P}$ be a basis for A. Then, $\{e \otimes e, e \otimes e_i, e_i \otimes e, e_i \otimes e_j\}_{i,j \in P}$ is a basis for $A \otimes A$. Applying $\bar{d}$ to $n(e \otimes e)$, we obtain that $\bar{d}(n)(e \otimes e) = 0$, since N is an ideal of annihilators. But then, if $\bar{d}(n) = \sum_i x_i \otimes y_i$ and $\sum_i x_i y_i = \alpha e + \sum_j \alpha_j e_j$, we have $0 = \sum_i x_i y_i \otimes e = \alpha e \otimes e + \sum_j \alpha_j e_j \otimes e$; it follows that $\alpha = \alpha_j = 0$ $(j \in P)$ and so $\sum_i x_i y_i = 0$. Therefore, $\bar{d}(n) \in N$. In the same way, we can prove that $\bar{\psi}(N) \subset N$ for any automorphism $\bar{\psi}$ of $A \otimes A$. ■

As is clear, similar results hold for the commutative duplicate of a commutative algebra.

The condition $A^2 = A$ in the proposition and in its corollary is essential, as shown by the following

EXAMPLE. Let K_n $(n > 1)$ be the real algebra with basis $c_0, c_1, \dots, c_n$ and multiplication table

$$c_0^2 = c_0, \qquad c_i c_j = 0 \quad \text{if } (i, j) \neq (0,0).$$

Notice that $K_n^2 \neq K_n$. For K_n, $\bar{d}(N) \subset N$ for any derivation $\bar{d}$ of $K_n \otimes K_n$, but the map $d \in \mathrm{Der}(K_n) \to d_\otimes \in \mathrm{Der}(K_n \otimes K_n)$ is not an isomorphism [5]. As is readily seen, the linear form $\rho \colon K_n \to \mathbf{R}$, defined by $\rho(c_0) = 1$, $\rho(c_i) = 0$ $(1 \leqslant i \leqslant n)$, preserves the multiplication of K_n, it is the unique nonzero linear form with this property, and for any $x, y \in K_n$, $xy = \rho(x)\rho(y)c_0$. If ψ is an automorphism of K_n, then $\rho\psi = \rho$ and so $\rho(x)\rho(y)[\psi(c_0) - c_0] = 0$ $\forall x, y \in K_n$; it follows that $\psi(c_0) = c_0$. On the other hand, any nonsingular linear operator ψ of K_n such that $\psi(c_0) = c_0$ is an automorphism. Therefore, the automorphism group of K_n is the general linear group $\mathrm{Gl}(n, \mathbf{R})$. The duplicate of K_n is isomorphic to $K_{n(n+2)}$, and its automorphism group is then $\mathrm{Gl}(n(n+2), \mathbf{R})$. Thus, for K_n, although $\bar{\psi}(N) \subset N$ for any automorphism $\bar{\psi}$ of $K_n \otimes K_n$, the map $\psi \in \mathrm{Aut}(K_n) \to \psi^\otimes \in \mathrm{Aut}(K_n \otimes K_n)$ is not an isomorphism.

3. APPLICATION

Let $G(n+1, 2m)$ be the gametic algebra for a $2m$-ploid population with $n+1$ alleles. As shown by Gonshor [3], this algebra has a basis consisting of all monomials $X_0^{m-p} X_{i_1} \cdots X_{i_p}$ of degree m in the variables $X_0, \dots, X_n$, and the multiplication table is given by

$$\left(X_0^{m-p}X_{i_1}\cdots X_{i_p}\right)\left(X_0^{m-q}X_{j_1}\cdots X_{j_q}\right)$$

$$=\binom{2m}{p+q}^{-1}\binom{m}{p+q}X_0^{m-(p+q)}X_{i_1}\cdots X_{i_p}X_{j_1}\cdots X_{j_q}\qquad \text{if}\quad p+q\leqslant m;$$

the other products are zero. The monomial X_0^m is an idempotent and $[G(n+1,2m)]^2=G(n+1,2m)$. The corresponding zygotic algebra $Z(n+1,2m)$ is the commutative duplicate of $G(n+1,2m)$.

THEOREM.

(i) *The automorphism group of* $Z(n+1,2m)$ *is isomorphic to the affine group of* $\mathbf{R}^n$.

(ii) *The derivation algebra of* $Z(n+1,2m)$ *is isomorphic to the Lie algebra* $\mathbf{R}^n\oplus \mathrm{gl}(n,\mathbf{R})$.

Proof. Part (i) follows from our corollary and Corollary 7 of [6]. Part (ii) is a consequence of part (i), since the Lie algebra of the automorphism group of an algebra is its algebra of derivations, and the Lie algebra of the affine group of $\mathbf{R}^n$ is $\mathbf{R}^n\oplus \mathrm{gl}(n,\mathbf{R})$. ■

Part (ii) of the preceding theorem was proved by Costa [5] by straightforward calculations on a canonical basis of $Z(n+1,2m)$.

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Received 6 May 1987; final manuscript accepted 22 July 1987