DERIVATIONS ON TERNARY RINGS OF OPERATORS

ROBERT PLUTA AND BERNARD RUSSO

ABSTRACT. To each projection p in a C^*-algebra A we associate a family of derivations on A, called *p-derivations*, and relate them to the space of triple derivations on $pA(1-p)$. We then show that every derivation on a ternary ring of operators is spatial and we investigate whether every such derivation on a weakly closed ternary ring of operators is inner.

1. S-DERIVATIONS ON C*-ALGEBRAS

If A is a C^*-algebra, we let $D(A)$ denote the Banach Lie algebra of derivations on A. To be more precise $D(A)$ consists of all operators $\delta \in B(A)$ that satisfy $\delta(xy) = \delta(x)y + x\delta(x)$ for every x, y in A. $B(A)$ denotes the bounded linear operators on A.

A derivation $\delta \in D(A)$ is called self-adjoint if $\delta = \delta^*$, where δ^* is the derivation defined by $\delta^*(x) = \delta(x^*)^*$ for every x in A. The space of all self-adjoint derivations on A is a real Banach Lie subalgebra of $D(A)$ and is denoted $D^*(A)$.

Derivations on C^*-algebras have suitable counterparts in a more general setting of ternary rings of operators, or TROs for short, where they are sometimes termed triple derivations. However, in this paper we shall use the term *triple derivation* to denote a derivation of a Jordan triple system. For example, if X is a Banach subspace of a C^*-algebra and $xy^*z + zy^*x \in X$ for every x, y, z in X, then X is called a JC*-triple and a *triple derivation* on X is an operator $\tau \in B(X)$ satisfying

$$\tau(\{xy^*z\}) = \{\tau(x)y^*z\} + \{x\tau(y)^*z\} + \{xy^*\tau(z)\}$$

for every x, y, z in X, where $\{xyz\} = (xy^*z + zy^*x)/2$.

We shall use the term TRO-derivation, as follows: If X is a Banach subspace of a C^*-algebra and $xy^*z \in X$ for every x, y, z in X, then X is called a TRO and a *TRO-derivation* on X is an operator $\tau \in B(X)$ satisfying

$$\tau(xy^*z) = \tau(x)y^*z + x\tau(y)^*z + xy^*\tau(z)$$

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for every x, y, z in X.

It is clear that a TRO (resp. JC*-triple) can also be defined as a Banach subspace of $B(H, K)$, the bounded operators from Hilbert space H to Hilbert space K, which is closed under the triple product xy^*z (resp. $(xy^*z + zy^*x)/2$). If a TRO is weakly closed, it is called a W*-TRO.

In this section we will introduce the class of S-derivations on a C^*-algebra A associated with a subspace $S \subseteq A$. Of particular interest will be the case $S = pAp$ for a projection p in A. We will seek to determine the relationship between the class of pAp derivations (which we call p-derivations for short) on A and the class of TRO-derivations on $pA(1-p)$.

Definition 1.1. Let A be a C^*-algebra and let S be a subspace of A. We say that a derivation $\delta \in D(A)$ is *associated with* S, or simply that δ is an S-*derivation*, if δ leaves S invariant in the sense that $\delta(S) \subseteq S$.

We use $D_S(A)$ to denote the set of all S-derivations. In order to simplify the notation, we write $D_e(A)$ for $D_{eAe}(A)$ in case $S = eAe$, for some idempotent $e \in A$, and we abuse the terminology slightly by referring to the elements of $D_e(A)$ simply as e-*derivations*.

To repeat, given an arbitrary idempotent e in a C^*-algebra A, which in particular may be a projection, by an e-derivation on A we mean a derivation $\delta \in D(A)$ satisfying $\delta(eAe) \subseteq eAe$. This condition is easily seen to be equivalent to the requirement that $\delta(e) = 0$.

Example 1.2. Let A be a C^*-algebra and let $e \in A$ be an idempotent. Fix $a \in eAe$ and $b \in (1-e)A(1-e) = \{x - xe - ex + exe : x \in A\}$. Then $\delta\colon A \to A$ defined by $\delta(x) = (a+b)x - x(a+b)$ is an e-derivation.

Lemma 1.3. *Let A be a C^*-algebra and let S be a subalgebra with an identity element 1_S (possibly different from the identity element of A if A is unital). Let $\delta \in D(A)$ be a derivation. The following statements hold.*

(1) If $\delta(S) \subseteq S$ then $\delta(1_S) = 0$.
(2) If $\delta(1_S) = 0$ then $\delta(S) \subseteq 1_S A 1_S$.

Proof. A straightforward consequence of the derivation property. □

Lemma 1.4. *Let A be a C^*-algebra and let $e \in A$ be an idempotent. Let $\delta \in D(A)$ be a derivation. The following statements hold.*

(1) *If $\delta(e) = 0$, then δ leaves invariant the following subspaces*

$$eAe, \quad eA(1-e), \quad (1-e)Ae, \quad (1-e)A(1-e).$$

(2) *If δ leaves invariant eAe or $(1-e)A(1-e)$, then $\delta(e) = 0$.*

Additionally, let $\delta = \delta^*$ and $e = e^*$. Then the following statement holds.

(3) *If δ leaves invariant $eA(1-e)$ or $(1-e)Ae$, then $\delta(e) = 0$.*

Proof. The assertions (1) and (2) are straightforward consequences of the derivation property. To prove (3), assume that $eA(1-e)$ is invariant for $\delta = \delta^*$, and $e = e^*$. Since $\delta(e) = \delta(e)e + e\delta(e)$, we have $e\delta(e)e = 0$ and hence

$$\delta(e) = e\delta(e)(1-e) + (1-e)\delta(e)e.$$

This shows that both $e\delta(e)$ and $\delta(e)(1-e)$ are equal to $e\delta(e)(1-e)$, and so both $e\delta(e)$ and $\delta(e)(1-e)$ are elements of the subspace $eA(1-e)$ which is invariant under δ.

We will show that $\delta(e) = 0$ by showing that $\delta(e)^2 = 0$. For this, we identify A with $\begin{pmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)A(1-e) \end{pmatrix}$ and write $\delta(e)$ and $\delta^2(e)$ as

$$\delta(e) = \begin{pmatrix} 0 & e\delta(e)(1-e) \\ (1-e)\delta(e)e & 0 \end{pmatrix}, \qquad \delta^2(e) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $\delta(e)^2 = \begin{pmatrix} e\delta(e)(1-e)\delta(e)e & 0 \\ 0 & (1-e)\delta(e)e\delta(e)(1-e) \end{pmatrix}$ and since

$$\begin{aligned} \delta(e\delta(e)) &= \begin{pmatrix} e\delta(e)(1-e)\delta(e)e+ea & eb \\ 0 & (1-e)\delta(e)e\delta(e)(1-e) \end{pmatrix} \in \begin{pmatrix} 0 & eA(1-e) \\ 0 & 0 \end{pmatrix}, \\ \delta(\delta(e)(1-e)) &= \begin{pmatrix} -e\delta(e)(1-e)\delta(e)e & b(1-e) \\ 0 & d(1-e)-(1-e)\delta(e)e\delta(e)(1-e) \end{pmatrix} \in \begin{pmatrix} 0 & eA(1-e) \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

it follows that $(1-e)\delta(e)e\delta(e)(1-e) = 0 = e\delta(e)(1-e)\delta(e)e$. Thus $\delta(e)^2 = 0$, as desired. □

If A is a C^*-algebra and $p \in A$ is a projection, we let $D_p^*(A)$ denote the (real) Banach Lie algebra of self-adjoint p-derivations on A. To be more precise $D_p^*(A)$ consists of all derivations $\delta \in D(A)$ that satisfy $\delta(p) = 0$ and $\delta = \delta^*$. If X is a TRO, we use $D_{TRO}(X)$ to denote the (real) Banach Lie algebra of all TRO-derivations on X.

Remark 1.5. Let A be a unital C^*-algebra and let $p \in A$ be a projection. Then the map

$$\Delta\colon D_p^*(A) \to D_{TRO}(pA(1-p)), \qquad \Delta(\delta) = \delta_{|pA(1-p)}$$

is a homomorphism of Banach Lie algebras.

Example 1.6. Let $A = M_2(\mathbb{C})$, $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The set of all p-derivations on A is:

$$\begin{aligned} D_p(A) &= \{\delta \in D(A) : \delta(p) = 0\} \\ &\simeq \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in \mathbb{C}\right\} \\ &= \text{a complex Banach Lie algebra.} \end{aligned}$$

The set of all self-adjoint p-derivations is:

$$\begin{aligned} D_p^*(A) &= \{\delta \in D_p(A) : \delta = \delta^*\} \\ &\simeq \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) : \alpha, \beta \in \mathbb{C} \text{ with } \Re(\alpha) = \Re(\beta) \right\} \\ &= \text{a real Banach Lie algebra.} \end{aligned}$$

The mapping

$$\Delta \colon D_p^*(A) \to D_{TRO}(X), \qquad \Delta(\delta) = \delta_{|X}$$

defines a linear surjection between the self-adjoint p-derivations on A and the TRO-derivations on $X = pA(1-p) = \left(\begin{smallmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{smallmatrix}\right)$. The kernel of Δ is isomorphic to the center of A, i.e.,

$$\ker \Delta = Z(A) = \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) : \alpha \in \mathbb{C} \right\}.$$

In other words, the TRO-derivations on $X = pA(1-p) = \left(\begin{smallmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{smallmatrix}\right)$ are precisely the self-adjoint p-derivations on the linking algebra $\left(\begin{smallmatrix} XX^* & X \\ X^* & X^*X \end{smallmatrix}\right) = A = M_2(\mathbb{C})$.

Example 1.7. Let $A = M_5(\mathbb{C})$, and let $p \in A$ be the projection matrix with 1 in the $(1,1)$ and $(2,2)$ position and zero's elsewhere. The set of all p-derivations on A is:

$$\begin{aligned} D_p(A) &= \{\delta \in D(A) : \delta(p) = 0\} \\ &\simeq \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) : A \in M_2(\mathbb{C}), B \in M_3(\mathbb{C}) \right\} \\ &= \text{a complex Banach Lie algebra.} \end{aligned}$$

The set of all self-adjoint p-derivations is $D_p^*(A) = \{\delta \in D_p(A) : \delta = \delta^*\}$ and it can be identified with the real Banach Lie algebra consisting of all matrices of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ where $A \in M_2(\mathbb{C})$, $B \in M_3(\mathbb{C})$, and $\left(\begin{smallmatrix} A+A^* & 0 \\ 0 & B+B^* \end{smallmatrix}\right)$ is in the center of A.

2. Derivations on TROs

If A is a unital C*-algebra and e is a projection in A, then $X := eA(1-e)$ is a TRO. Conversely if $X \subset B(K,H)$ is a TRO, then with $X^* = \{x^* : x \in X\} \subset B(H,K)$, $XX^* = \operatorname{span}\{xy^* : x, y \in X\} \subset B(H)$, $X^*X = \operatorname{span}\{z^*w : z, w \in X\} \subset B(K)$, $K_l(X) = \overline{XX^*}^n$, $K_r(X) = \overline{X^*X}^n$, we let[1]

$$A_X = \begin{bmatrix} K_l(X) + \mathbb{C}1_H & X \\ X^* & K_r(X) + \mathbb{C}1_K \end{bmatrix} \subset B(H \oplus K)$$

[1]If $K_l(X)$ and $K_r(X)$ are unital subalgebras of $B(H)$ and $B(K)$ (resp.), and X is nondegenerate, that is, XX^*X is dense in X, then we take A_X to be $\left[\begin{smallmatrix} K_l(X) & X \\ X^* & K_r(X) \end{smallmatrix}\right]$

denote the (unital) linking C*-algebra of X. Then we have a TRO-isomorphism $X \simeq eA_X(1-e)$, where $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Lemma 2.1. *Let X be a TRO and let $D : X \to X$ be a TRO-derivation of X. If $A_0 = \left(\begin{smallmatrix} XX^* & X \\ X^* & X^*X \end{smallmatrix}\right)$, then the map $\delta_0 : A_0 \to A_0$ given by*

$$\begin{pmatrix} \sum_i x_i y_i^* & x \\ y^* & \sum_j z_j^* w_j \end{pmatrix} \mapsto \begin{pmatrix} \sum_i (x_i (Dy_i)^* + (Dx_i) y_i^*) & Dx \\ (Dy)^* & \sum_j (z_j^*(Dw_j) + (Dz_j)^* w_j) \end{pmatrix}$$

*is well defined and a bounded *-derivation of A_0, which extends D (when X is embedded in A_X via $x \mapsto \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$), and which itself extends to a *-derivation δ of A_X. Thus, the Lie algebra homomorphism $\Delta : \delta \mapsto \delta_{|X}$ given in Remark 1.5 is onto.*

Proof. If $\sum_i x_i y_i^* = 0$, then for every $z \in X$,

$$\begin{aligned} 0 &= D(\sum_i x_i y_i^* z) \\ &= \sum_i ((Dx_i) y_i^* z + x_i (Dy_i)^* z + x_i y_i^* (Dz)) \\ &= (\sum_i ((Dx_i) y_i^* + x_i (Dy_i)^*)) z. \end{aligned}$$

Since this is true for every z, we have $\sum_i ((Dx_i) y_i^* + x_i (Dy_i)^*) = 0$ (see [4, Lemma 2.3(iv)]) and it follows that δ_0 is well defined.

The map δ_0 is self-adjoint since if $a = \left(\begin{smallmatrix} \sum_i x_i y_i^* & x \\ y^* & \sum_j z_j^* w_j \end{smallmatrix}\right)$, then

$$\begin{aligned} \delta_0(a^*) &= \delta_0 \begin{pmatrix} \sum_i y_i x_i^* & y \\ x^* & \sum_j w_j^* z_j \end{pmatrix} \\ &= \begin{pmatrix} \sum_i (y_i (Dx_i)^* + (Dy_i) x_i^*) & Dy \\ (Dx)^* & \sum_j (w_j^* (Dz_j) + (Dw_j)^* z_j) \end{pmatrix} \\ &= \delta_0(a)^*. \end{aligned}$$

It is easy to verify that $\delta_0(a^2) = \delta_0(a)a + a\delta_0(a)$ so that δ_0 is a Jordan *-derivation of A_0. (We omit that calculation.)

To see that δ_0 is bounded, we first note that D is bounded, since it is a Jordan triple derivation on the JB*-triple X with the Jordan triple product $\{xyz\} = (xy^*z + zy^*x)/2$, and hence bounded by the theorem of Barton and Friedman [1]. Now denoting $\sum_i (x_i (Dy_i)^* + (Dx_i) y_i^*)$ by

α, we have (by [4, Lemma 2.3(iv)] again) $\|\sum_i (x_i(Dy_i)^* + (Dx_i)y_i^*)\| =$

$$\begin{aligned}
\|\alpha\| &= \sup_{\|z\|\le 1, z\in X} \|\alpha z\| \\
&= \sup_{\|z\|\le 1, z\in X} \|\alpha z + \overbrace{\sum_i x_i y_i^*(Dz) - \sum_i x_i y_i^*(Dz)}^{=0}\| \\
&= \sup_{\|z\|\le 1, z\in X} \|\sum_i D(x_i y_i^* z) - \sum_i x_i y_i^*(Dz)\| \\
&= \sup_{\|z\|\le 1, z\in X} \|D\sum_i x_i y_i^* z - \|D\| \sum_i x_i y_i^* \frac{Dz}{\|D\|}\| \\
&\le 2\|D\| \|\sum_i x_i y_i^*\|.
\end{aligned}$$

Thus δ_0 is bounded and therefore extends to a bounded Jordan *-derivation δ of $\overline{A_0}^n$ and hence to A_X by setting $\delta(e) = 0$, where $e = \left[\begin{smallmatrix} 1_H & 0 \\ 0 & 0 \end{smallmatrix}\right]$. By the theorem of Sinclair ([15, Theorem 3.3]), δ is a derivation of A_X.[2] □

For any C*-algebra $A \subset B(H)$, the Lie algebra homomorphism $\overline{A}^w \ni z \mapsto \operatorname{ad} z \in D(\overline{A}^w)$ is onto (theorem of Kadison and Sakai ([14, 4.1.6])) and so we have the Lie algebra isomorphism

$$\overline{A}^w / Z(\overline{A}^w) \simeq D(\overline{A}^w).$$

It follows (cf. [14, 4.1.7]) that

$$\{t \in \overline{A}^w : \operatorname{ad} t(A) \subset A\}/Z(\overline{A}^w) \simeq D(A),$$

and

$$\{t \in \overline{A}^w : t^* = -t, \operatorname{ad} t(A) \subset A\}/Z(\overline{A}^w) \simeq D^*(A).$$

Further, for a projection e in A, we have

$$\{t \in \overline{A}^w : et = te, t^* = -t, \operatorname{ad} t(A) \subset A\}/Z(\overline{A}^w) \simeq D_e^*(A).$$

Using these facts in the setting of Lemma 2.1, and noting that, by [9, page 268], $\overline{A_X}^w = A_X'' = \left[\begin{smallmatrix} K_l(X)'' & \overline{X}^w \\ \overline{X^*}^w & K_r(X)'' \end{smallmatrix}\right]$, we can now prove the following theorem.

Theorem 2.2. *Every TRO-derivation of a TRO X is spatial in the sense that there exist $\alpha \in K_l(X)''$ and $\beta \in K_r(X)''$ such that $\alpha^* = -\alpha$, $\beta^* = -\beta$, and $Dx = \alpha x + x\beta$ for every $x \in X$.*

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[2] It is also easy to verify directly, by (a more involved) calculation, that δ_0 is a derivation, thereby avoiding the use of Sinclair's theorem

Proof. If $D \in D_{TRO}(X)$, choose $\delta = \operatorname{ad} t$ for some $t \in \overline{A_X}^w$ with $t^* = -t$, $te = et$ and

$$\begin{bmatrix} 0 & Dx \\ 0 & 0 \end{bmatrix} = \delta \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}.$$

The conditions on t imply that $t = \left[\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right]$ with $\alpha^* = -\alpha$ and $\beta^* = -\beta$. Moreover

$$\delta \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} 0 & \alpha x + x(-\beta) \\ 0 & 0 \end{bmatrix}.$$

□

A TRO derivation D of a TRO X is said to be an *inner TRO derivation* if there exist $\alpha = -\alpha^* \in XX^*$ and $\beta = -\beta^* \in X^*X$ such that $Dx = \alpha x + x\beta$ for $x \in X$. Note that there exist $a_i, b_i, c_j, d_j \in X$, $1 \leq i \leq n, 1 \leq j \leq m$ such that $\alpha = \sum_{i=1}^n (a_i b_i^* - b_i a_i^*)$ and $\beta = \sum_{j=1}^m (c_j^* d_j - d_j^* c_j)$.

Corollary 2.3. *Every TRO derivation of a C^*-algebra A is of the form $A \ni x \mapsto \alpha x + x\beta$ with elements $\alpha, \beta \in \overline{A}^w$ with $\alpha^* = -\alpha, \beta^* = -\beta$. In particular, every TRO derivation of a von Neumann algebra is an inner TRO derivation*

Thus, every W*-TRO which is TRO-isomorphic to a von Neumann algebra has only inner TRO derivations. For example, this is the case for the stable W*-TROs of [12] (see subsection 3.2) and the weak*-closed right ideals in certain continuous von Neumann algebras acting on separable Hilbert spaces (see Theorem 3.3).

Theorem 2.2 is an improvement of [17], in which, although proved for the slightly more general case of derivation pairs, it is assumed that the TRO (called B*-triple system in [17]) contains the finite rank operators. For the extension of Zalar's result to unbounded operators, see [16].

A triple derivation δ of a JC*-triple X is said to be an *inner triple derivation* if there exist finitely many elements $a_i, b_i \in X$, $1 \leq i \leq n$, such that $\delta x = \sum_{i=1}^n (\{a_i b_i x\} - \{b_i a_i x\})$ for $x \in X$, where $\{xyz\} = (xy^*z + zy^*x)/2$. For convenience, we denote the inner triple derivation $x \mapsto \{abx\} - \{bax\}$ by $\delta(a, b)$. Thus

$$\delta(a, b)(x) = (ab^*x + xb^*a - ba^*x - xa^*b)/2.$$

Let X be a TRO. As noted in the proof of Lemma 2.1, X is a JC*-triple in the triple product $(xy^*z + zy^*x)/2$, and every TRO-derivation of X is obviously a triple derivation. On the other hand, every inner triple derivation is an inner TRO-derivation. Indeed, if

$\delta(x) = \{abx\} - \{bax\}$, for some $a, b \in X$, then $\delta(x) = Ax + xB$, where $A = ab^* - ba^* \in XX^*$, $B = b^*a - a^*b \in X^*X$ with A, B skew-hermitian. Moreover, since by [1, Theorem 4.6], every triple derivation δ on X is the strong operator limit of a net δ_α of inner triple derivations, hence TRO-derivations, we have (i) and (ii) in the following proposition.

Proposition 2.4. *Let X be a TRO.*

- **(i):** *Every TRO-derivation is the strong operator limit of inner TRO-derivations.*
- **(ii):** *The triple derivations on X coincide with the TRO-derivations.*
- **(iii):** *The inner triple derivations on X coincide with the inner TRO-derivations*
- **(iv):** *All TRO derivations of X are inner, if and only if, all triple derivations of X are inner.*

Proof. Since (iv) is immediate from (ii) and (iii), we only need to show part of (iii), that is, that every inner TRO-derivation is an inner triple derivation. If D is an inner TRO-derivation, then $Dx = \alpha x + x\beta$, with $\alpha^* = -\alpha \in XX^*$ and $\beta^* = -\beta \in X^*X$. We must show that there exist elements a_k, b_k such that $Dx = \sum_{k=1}^{p} \delta(a_k, b_k)x$ where $\delta(a_k, b_k)$ is the inner triple derivation $x \mapsto \{a_k b_k x\} - \{b_k a_k x\}$. If $\alpha = \sum_{i=1}^{n} x_i y_i^*$ and $\beta = \sum_{j=1}^{m} z_j^* w_j$, then it suffices to take $p = m + n$ and choose $a_i = x_i/2$, $b_i = y_i$ for $1 \leq i \leq n$ and $a_{n+i} = w_i$, $b_{n+i} = z_i/2$ for $1 \leq i \leq m$. □

3. Derivations on W*-TROs

A von Neumann algebra M is an example of a unital reversible JW^*-algebra, and as such, by [6, Theorem 2 and the first sentence in its proof], every triple derivation on M is an inner triple derivation. Hence we see that the last statement in Corollary 2.3 follows also from this and Proposition 2.4(iv). For completeness, we include a proof of the former result which avoids much of the Jordan theory, starting with the following lemma, the first part of which is straightforward.

Lemma 3.1. *Let A be a unital Banach *-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ and the Jordan product $a \circ b = (ab + ba)/2$.*

- *Let D be an inner derivation, that is, $D =$ ad $a : x \mapsto ax - xa$, for some a in A. Then $D =$ ad a is a *-derivation whenever $a^* = -a$. Conversely, if D is a *-derivation, then $a^* = -a + z$ for some z in the center of A.*
- *Every triple derivation is the sum of a Jordan *-derivation and an inner triple derivation.*

Proof. To prove the second statement, we modify the proof in [7, Section 3] which is in a different context. We note first that for a triple derivation δ, $\delta(1)^* = -\delta(1)$. Next, for a triple derivation δ, the mapping $\delta_1(x) = \delta(1) \circ x$ is equal to the inner triple derivation $-\frac{1}{4}\delta(\delta(1), 1)$ so that $\delta_0 := \delta - \delta_1$ is a triple derivation with $\delta_0(1) = 0$. Finally, any triple derivation which vanishes at 1 is a Jordan *-derivation. □

Theorem 3.2. *Every triple derivation on a von Neumann algebra is an inner triple derivation.*

Proof. It suffices, by the second statement in Lemma 3.1, to show that every self-adjoint Jordan derivation is an inner triple derivation. If δ is a self-adjoint Jordan derivation of M, then δ is an associative derivation (by the theorem of Sinclair, [15, Theorem 3.3])) and hence by the theorem of Kadison and Sakai ([14, 4.1.6]) and the first statement in Lemma 3.1, $\delta(x) = ax - xa$ where $a^* + a = z$ is a self adjoint element of the center of M. Since for every von Neumann algebra, we have $M = Z(M) + [M, M]$, where $Z(M)$ denotes the center of M (see [11, Section 3] for a discussion of this fact), we can therefore write

$$\begin{aligned} a &= z' + \sum_j [b_j + ic_j, b'_j + ic'_j] \\ &= z' + \sum_j ([b_j, b'_j] - [c_j, c'_j]) + i \sum_j ([c_j, b'_j] + [b_j, c'_j]), \end{aligned}$$

where b_j, b'_j, c_j, c'_j are self adjoint elements of M and $z' \in Z(M)$.

It follows that

$$0 = a^* + a - z = (z')^* + z' - z + 2i \sum_j ([c_j, b'_j] + [b_j, c'_j])$$

so that $\sum_j ([c_j, b'_j] + [b_j, c'_j])$ belongs to the center of M. We now have

$$\delta = \operatorname{ad} a = \operatorname{ad} \sum_j ([b_j, b'_j] - [c_j, c'_j]).$$

A direct calculation shows that δ is equal to the inner triple derivation $\sum_j \left(\delta(b_j, 2b'_j) - \delta(c_j, 2c'_j)\right)$, completing the proof. □

3.1. **Weakly closed right ideals in von Neumann algebras.** In this subsection, we shall consider the TRO pM where M is a von Neumann algebra and p is a projection in M.

A TRO of the form pM, with M a continuous von Neumann algebra, is classified into four types in [8] as follows.

- II_1^a if M is of type II_1 and p is (necessarily) finite.
- $II_{\infty,1}^a$ if M is of type II_∞ and p is a finite projection.

- II^a_∞ if M is of type II_∞ and p is a properly infinite projection.
- III^a if M is of type III and p is a (necessarily) properly infinite projection.

Similarly, we also define types for pM for M of type I:

- I^a_1 if M is finite of type I and p is (necessarily) finite.
- $I^a_{\infty,1}$ if M is of type I_∞ and p is a finite projection.
- I^a_∞ if M is of type I_∞ and p is a properly infinite projection.

The following theorem involves the cases II^a_∞, III^a and when M is a factor, the cases $I^a_1, I^a_{1,\infty}$, and I^a_∞.

Theorem 3.3. *Let $X = pM$ be a TRO, where M is a von Neumann algebra and p is a projection in M.*

- **(i):** *If X is of type II^a_∞ or III^a, and has a separable predual, then every TRO-derivation of X is an inner TRO-derivation.*
- **(ii):** *If M is of type III and countably decomposable, then every TRO-derivation of $X = pM$ is an inner TRO-derivation.*
- **(iii):** *If $M = B(H)$ is a factor of type I, then*
  1. *If $\dim H < \infty$, then every TRO-derivation of $X = pM$ is an inner TRO-derivation.*
  2. *If $\dim pH = \dim H$, then every TRO-derivation of $X = pM$ is an inner TRO-derivation.*
  3. *If $\dim pH < \dim H = \infty$, then $X = pM$ admits outer TRO-derivations.*

Proof. If M is a continuous von Neumann algebra with a separable predual and p is a properly infinite projection in M, then it is shown in [8, Theorem 5.16] that pM is triple isomorphic to a von Neumann algebra, and hence by Theorem 3.2, every triple derivation is an inner triple derivation in this case. Consequently, by Proposition 2.4(iv), every TRO-derivation is an inner TRO-derivation. (Another way to see this latter fact is to note that by [8, Lemma 5.15], pM is actually TRO-isomorphic to a von Neumann algebra, and to apply Corollary 2.3.) This proves (i).

To prove (ii), we note first that if A is a von Neumann algebra with a projection $p \sim 1$, then pA is TRO-isomorphic to A. Indeed, If u is a partial isometry in A with $uu^* = p$ and $u^*u = 1$, then $x \mapsto u^*x$ is a TRO-isomorphism from pA onto A. Now if A is of type III, then $\tilde{A} := c(p)A$ is of type III, $c(p)$ is the identity of $\tilde{A}$ and $pA = p\tilde{A}$. Further, if A is countably decomposable, then by [14, 2.2.14], since in $\tilde{A}$, $c(p) = 1_{\tilde{A}} = c(1_{\tilde{A}})$, we have $p \sim 1_{\tilde{A}}$, so $\tilde{A}$ is TRO-isomorphic to $p\tilde{A} = pA$.

Finally, let $M = B(H)$. The first statement in (iii) follows from the fact that every finite dimensional semisimple Jordan triple system has only inner derivations. This result first appeared in [10, Chapter 11] (see also [13, Theorem 2.8,p. 136]). If $\dim pH = \dim H$, then $pM \simeq B(H)$ has only inner triple derivations by Theorem 3.2. On the other hand, if $\dim pH < \dim H = \infty$, then $pM \simeq B(H, pH)$ has outer triple derivations, as shown in [6, Corollary 3]. By Proposition 2.4(iv), this proves (iii) □

Remark 3.4. Although it follows from Theorem 3.3, it is worth pointing out that the TROs $B(\mathbb{C}, H)$ and $B(H, \mathbb{C})$ support outer TRO derivations if and only if $\dim H = \infty$. According to [8, Lemma 5.15], if B is a von Neumann algebra of type II_∞ or III, and H is a separable Hilbert space, then B and $B\overline{\otimes}B(\mathbb{C}, H)$ are TRO-isomorphic. Corollary 2.3 shows that $B\overline{\otimes}B(\mathbb{C}, H)$ has only inner TRO-derivations and only inner triple derivations, although, as just noted, $B(\mathbb{C}, H)$ can have an outer TRO derivation and an outer triple derivation. This contrasts the situation of derivations on tensor products of C*-algebras, as in [2, Proposition 3.2]).

3.2. **W*-TROs of types I,II,III.** We begin by recalling some concepts from [12]. If R is a von Neumann algebra and e is a projection in R, then $V := eR(1-e)$ is a W*-TRO. Conversely if $V \subset B(K, H)$ is a W*-TRO, then with $V^* = \{x^* : x \in V\} \subset B(H, K)$, $M(V) = \overline{XX^*}^{sot} \subset B(H)$, $N(V) = \overline{X^*X}^{sot} \subset B(K)$, we let

$$R_V = \begin{bmatrix} M(V) & V \\ V^* & N(V) \end{bmatrix} \subset B(H \oplus K)$$

denote the linking von Neumann algebra of V. Then we have a SOT-continuous TRO-isomorphism $V \simeq eRe^\perp$, where $e = \left[\begin{smallmatrix} 1_H & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $e^\perp = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1_K \end{smallmatrix}\right]$.

A W*-TRO V is *stable* if it is TRO-isomorphic to $B(\ell_2)\overline{\otimes}V$. A W*-TRO is of type I,II,or III, by definition, if its linking von Neumann algebra is of that type as a von Neumann algebra. There is a further classification of the types I and II depending on the types of $M(V)$ and $N(V)$ leading to the types $I_{m,n}, II_{\alpha,\beta}$ where m, n are cardinal numbers and $\alpha, \beta \in \{1, \infty\}$. See [12, Section 4] for detail.

In what follows, for ultraweakly closed subspaces $A \subset M$ and $B \subset N$, where M and N are von Neumann algebras, $A\overline{\otimes}B$ denotes the ultraweak closure of the algebraic tensor product $A \otimes B$.

We shall use the following results from [12], which we summarize as a theorem.

Theorem 3.5 (Ruan [12]). *Let V be a W*-TRO acting on separable Hilbert spaces.*

(i) [12, Theorem 3.2] *If V is a stable W*-TRO, then V is TRO-isomorphic to $M(V)$ and to $N(V)$.*

(ii) [12, Corollary 4.3] *If V is a W*-TRO of one of the types $I_{\infty,\infty}, II_{\infty,\infty}$ or III, then V is a stable W*-TRO, and hence TRO-isomorphic to a von Neumann algebra.*

(iii) [12, Theorem 4.4] *If V is a W*-TRO of type $II_{1,\infty}$ (respectively $II_{\infty,1}$), then V is TRO-isomorphic to $B(H,\mathbb{C})\overline{\otimes}M$ (respectively $B(\mathbb{C},H)\overline{\otimes}N$), where M (respectively N) is a von Neumann algebra of type II_1.*

Because taking a transpose is a triple isomorphism, we have the following consequence of Theorem 3.5(iii).

Lemma 3.6. *A W*-TRO of type $II_{1,\infty}$ is triple isomorphic to a W*-TRO of type $II_{\infty,1}$. More precisely, $B(H,\mathbb{C})\overline{\otimes}M$ is triple isomorphic to $B(\mathbb{C},H)\overline{\otimes}M^t$, where $x^t = Jx^*J$, for $x \in M \subset B(H)$ and J is a conjugation on H.*

Proposition 3.7. *Let V be a W*-TRO.*

- **(i):** *If V acts on a separable Hilbert space and is of one of the types $I_{\infty,\infty}, II_{\infty,\infty}$ or III, then every triple derivation of V is an inner triple derivation and every TRO derivation of V is an inner TRO-derivation.*
- **(ii):** *If every TRO-derivation of any W*-TRO of type $II_{1,\infty}$ has only inner TRO-derivations, then every TRO-derivation of any W*-TRO of type $II_{\infty,1}$ has only inner TRO-derivations. The converse also holds.*

Proof. (i) is an immediate consequence of Theorem 3.5(ii), Theorem 3.2 and Proposition 2.4(iv). (ii) is an immediate consequence of Lemma 3.6 and Proposition 2.4(iv). □

It follows from Remark 3.4 that if M is a von Neumann algebra of type II_∞ or III and H is a separable Hilbert space, then $B(\mathbb{C},H)\overline{\otimes}M$ is triple isomorphic to a von Neumann algebra and hence has only inner TRO-derivations, giving alternate proofs of parts of Proposition 3.7(i).

By [12, Theorem 4.1], if V is a W*-TRO of type I, then V is TRO-isomorphic to $\oplus_\alpha L^\infty(\Omega_\alpha)\overline{\otimes}B(K_\alpha,H_\alpha)$. In the next two results, we consider the related TRO $C(\Omega,B(H,K))$, where Ω is a compact Hausdorff space.

Lemma 3.8. *Let E be a TRO and Ω a compact Hausdorff space.*

(i): *If every TRO derivation of* $V := C(\Omega, E)$ *is an inner TRO derivation, then the same holds for* E.

(ii): *If every triple derivation of* $V := C(\Omega, E)$ *is an inner triple derivation, then the same holds for* E.

Proof. By Proposition 2.4(iv), it is sufficient to prove (i).

If D is a TRO derivation of E, then $\delta f(\omega) := D(f(\omega))$ is a TRO derivation of V, as is easily checked. Suppose every TRO derivation of V is an inner TRO derivation. Then $\delta f = \alpha f + f\beta$, where $\alpha = -\alpha^* = \sum_i x_i y_i^*$ for some $x_i, y_i \in V$, and $\beta = -\beta^* = \sum_i z_j^* w_j$ for some $z_j, w_j \in V$.

For $a \in E$, let $1 \otimes a \in V$ be the constant function equal to a. Then $D(a) = D((1 \otimes a)(\omega)) = \delta(1 \otimes a)(\omega) = \alpha(\omega)a + a\beta(\omega)$ for all $\omega \in \Omega$. Since $\alpha(\omega)^* = -\alpha(\omega) \in EE^*$ and $\beta(\omega)^* = -\beta(\omega) \in E^*E$, D is an inner TRO derivation of E. ☐

Recall from Theorem 3.3(iii) that the TRO $B(H, K)$ supports outer TRO derivations if and only if it is infinite dimensional and $\dim H \neq \dim K$.

Proposition 3.9. *If* $V = \oplus_\alpha C(\Omega_\alpha, E_\alpha)$*, where* $E_\alpha = B(K_\alpha, H_\alpha)$ *and if every triple derivation of* V *is an inner triple derivation, then for every* α*, either* $\dim E_\alpha < \infty$ *or* $\dim K_\alpha = \dim H_\alpha$.

Proof. Let δ be a triple derivation of V, and let $\delta_\alpha = \delta|_{C(\Omega_\alpha, E_\alpha)}$, which is a triple derivation of the weak*-closed ideal $C(\Omega_\alpha, E_\alpha)$. Then $\delta(\{f_\alpha\}) = \{\delta_\alpha f_\alpha\}$. Moreover if δ is an inner triple derivation, say $\delta = \sum_i \delta(a^i, b^i)$ for $a^i = \{a^i_\alpha\}, b^i = \{b^i_\alpha\} \in V$, then $\delta_\alpha = \sum_i \delta(a^i_\alpha, b^i_\alpha)$ is an inner triple derivation of $C(\Omega_\alpha, E_\alpha)$.

Now suppose every that every triple derivation of V is an inner triple derivation, and that for some α_0, E_{α_0} is infinite dimensional and $\dim K_{\alpha_0} \neq \dim H_{\alpha_0}$. Then, as noted above, there is an outer triple derivation D of $C(\Omega_{\alpha_0}, E_{\alpha_0})$. Then the triple derivation on V which is zero on $C(\Omega_\alpha, E_\alpha)$ for $\alpha \neq \alpha_0$ and equal to D on $C(\Omega_{\alpha_0}, E_{\alpha_0})$, cannot be inner by the preceding paragraph, which is a contradiction. ☐

4. Some questions left open

Questions 1. It remains to complete the results of Theorem 3.3 to include the cases where p is a finite projection in a continuous von Neumann algebra, or when p is arbitrary and M is a general von Neumann algebra of type I. As a possible tool for the first question, we note that there is an alternate proof of Proposition 2.4 (ii), in the case $X = pM$, p finite, using the technique in [5, Section II.B].

Questions 2. Besides the problem of extending the known cases to non separable Hilbert spaces, the cases left open in Proposition 3.7 for arbitrary W*-TROs are those of types $II_{1,1}$ and $II_{1,\infty}$ (the latter being equivalent to $II_{\infty,1}$).

Questions 3. Let E be a W*-TRO, and let $V = \oplus_\alpha L^\infty(\Omega_\alpha)\overline{\otimes}B(K_\alpha, H_\alpha)$ be a W*-TRO of type I.

- If every derivation of the W*-TRO $L^\infty(\Omega)\overline{\otimes}E$ is inner, does it follow that every derivation of E is inner?
- If every derivation of V is inner, does it follow that $\dim B(K_\alpha, H_\alpha) < \infty$, for all α; or $\sup_\alpha \dim B(K_\alpha, H_\alpha) < \infty$?
- If $\sup_\alpha \dim B(K_\alpha, H_\alpha) < \infty$, does it follow that every derivation of V is inner?

Remark 4.1. With respect to Questions 3,

(i) In the first bullet, if E had a separable predual, then a variant of [14, 1.22.13] would state that $L^\infty(\Omega)\overline{\otimes}E = L^\infty(\Omega, E)$ and the technique in Lemma 3.8 could be used.

(ii) In the first bullet, suppose that $E = pM$, with M a von Neumann algebra in $B(H)$ and p a projection in M, and let D is a derivation of E. Then $\delta := id \otimes D$ is a derivation of $V = L^\infty(\Omega)\overline{\otimes}E$. Assuming that δ is inner, there exist $\alpha = -\alpha^* \in VV^* = L^\infty(\Omega)\overline{\otimes}(\overline{EE^*})$ ($\overline{EE^*}$ denoting the weak closure) and $\beta \in V^*V = L^\infty(\Omega)\overline{\otimes}(\overline{E^*E})$, such that

$$1 \otimes Dx = \alpha(1 \otimes x) + (1 \otimes x)\beta, \quad (x \in E).$$

We have $EE^* = pMp \subset B(pH)$, $\overline{E^*E} \subset B(H)$, and $L^\infty(\Omega) \subset B(L^2(\Omega))$. For each $\varphi \in B(L^2(\Omega))_*$, let $R_\varphi : B(L^2(\Omega) \otimes pH) \to B(pH)$ be the slice map of Tomiyama defined by $R_\varphi(f \otimes x) = \varphi(f)x$ ([3, Lemma 7.2.2]).

Since VV^* is the ultraweak closure of $L^\infty(\Omega) \otimes EE^*$, by the weak*-continuity of R_φ, we have

$$\varphi(1)Dx = R_\varphi(\alpha)x + xR_\varphi(\beta)$$

with $R_\varphi(\alpha) \in EE^*$ and $R_\varphi(\beta) \in \overline{E^*E}$. Thus, if $\dim H = \dim pH$, or if E is finite dimensional, then $R_\varphi(\beta) \in E^*E$, so that D is an inner TRO-derivation, (take φ to be a normal state so that R_φ is self-adjoint and $R_\varphi(\alpha)^* = -R_\varphi(\alpha)$ and $R_\varphi(\beta)^* = -R_\varphi(\beta)$.) In general, D could be called a "quasi-inner" TRO-derivation.

(iii) In the second bullet, if each $B(K_\alpha, H_\alpha)$ had a separable predual, then a variant of [14, 1.22.13] would state that $L^\infty(\Omega_\alpha)\overline{\otimes}B(K_\alpha, H_\alpha) = L^\infty(\Omega_\alpha, B(K_\alpha, H_\alpha))$ and the technique in Proposition 3.9 could be used.

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Department of Mathematics, University of California, Irvine, CA 92697-3875, USA

E-mail address: `plutar@tcd.ie`

Department of Mathematics, University of California, Irvine, CA 92697-3875, USA

E-mail address: `brusso@math.uci.edu`