ANTI-C*ALGEBRAS

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Abstract. We introduce a class of Banach algebras that we call anti-C*-algebras. We show that the normed standard embedding of a C*-ternary ring is the direct sum of a C*-algebra and an anti-C*-algebra. We prove that C*-ternary rings and anti-C*-algebras are semisimple. We give two new characterizations of C*-ternary rings which are isomorphic to a TRO (ternary ring of operators), providing answers to a query raised by Zettl in 1983, and we propose some problems for further study.

1. Introduction

One of the results of [15], namely, [15, Prop. 2.7], is not accurately formulated as stated. It is not true that the normed standard embedding $\mathcal{A}(M)$ of a C*-ternary ring M is always a C*-algebra. The corrected version of Proposition 2.7 in [15] states that $\mathcal{A}(M)$ is the direct sum of a C*-algebra (corresponding to the positive sub-C*-ternary ring M_+ of M) and a Banach algebra $\mathcal{B}$ (corresponding to the negative sub-C*-ternary ring M_- of M). This is made precise in Theorem 2.7 below. The algebra $\mathcal{B}$ is a Banach algebra with a continuous involution and an approximate identity, and exploring the properties of this class of algebras is a topic worthy of further study. We embark on that study in this paper by showing, among other things, that it is semisimple.

This correction affects only the results of [15, Section 4]. The necessary adjustments for [15, Section 4] are provided in detail in section 5 below. All other results of [15] remain valid as stated.

In this paper we shall call the algebra $\mathcal{B}$ an *anti-C*-algebra*, since on the one hand, it is derived from an anti-TRO in the same way that the linking C*-algebra is derived from a TRO (Definition 2.3). Another justification is that, although by virtue of the semisimplicity it has a unique complete norm topology, it cannot be renormed to be a C*-algebra.

In section 2, we introduce the algebra $\mathcal{B}$ via a dense *-subalgebra $\mathcal{B}_0$ (Proposition 2.2), and in addition to Theorem 2.7, we give two new characterizations of when a C*-ternary ring is isomorphic to a TRO (Theorem 2.8). In section 3 we prove the semisimplicity of $\mathcal{B}$ (Theorem 3.9). In section 4 we state a relation between the ideals of a C*-ternary ring and those of its normed standard embedding (Proposition 4.2), and include a list of questions for further study. In section 5, as noted above, we revisit the topic of ternary operator categories, indicating the necessary adjustments to [15, Section 4]. An appendix reviews the construction of the standard embedding of an associative triple system.

Two distinct kinds of associative triple systems are recognized in the literature: those of the first kind and those of the second kind. The first one has been studied by Lister [10] but their mention here is solely for the purpose of completeness and will not be exploited. Instead, our attention will be directed toward examining the latter. An associative triple system of the second kind is defined as a complex vector space M equipped with a function $(x, y, z) \mapsto [xyz]$ from $M \times M \times M$ to M that exhibits linearity with respect to the outer variables, conjugate-linearity with respect to the middle variable, and satisfies

$$[[xyz]uv] = [x[uzy]v] = [xy[zuv]]$$

for all x, y, z, u, v in M.

These systems have been studied by Hestenes [7], Loos [11], Meyberg [12], and others, for example, [1, 9, 13–15, 19]. Recall that a *C*-ternary ring* was introduced by Zettl[1] in [19] as an associative triple

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[1]Which he called a *ternary C*-ring*

system M which is also a complex Banach space for which

$$\|[x,y,z]\| \leq \|x\|\|y\|\|z\| \text{ and } \|[x,x,x]\| = \|x\|^3.$$

In addition, if M is a dual Banach space, it is called a *W^*-ternary ring*.

In what follows, we shall use some notation and some results from [15], making precise references to [15] when necessary.

2. The normed standard embedding of a C*-ternary ring

We begin by recalling the construction of the linking algebra of a TRO. This will motivate our introduction of an anti-C*-algebra.

Remark 2.1. Let $M \subset B(H,K)$ be a ternary ring of operators ([15, Subsection 1.3]), so that M is a C*-ternary ring with the triple product $[xyz] := xy^*z$. Then the standard embedding $\mathcal{A}_0(M)$ of M is a pre-C*-algebra, which is *-isomorphic to a dense *-subalgebra of the linking C*-algebra A_M of M ([15, Example 1.6]). See [15, Subsection 1.2] for the construction of $\mathcal{A}_0(M)$, denoted there by $\mathcal{A}(M)$. (See the appendix to this paper for a summary of the construction.) In this paper we use $\mathcal{A}(M)$ to denote the normed standard embedding of a C*-ternary ring ([15, Remark 1.4]).

Recall that A_M is the closure of

$$\begin{bmatrix} MM^* & M \\ M^* & M^*M \end{bmatrix} = \left\{ \begin{bmatrix} \sum_i x_i y_i^* & z \\ w^* & \sum_j u_j^* v_j \end{bmatrix} : x_i, y_i, u_j, v_j, z, w \in M \right\} \subset \begin{bmatrix} B(K) & B(H,K) \\ B(K,H) & B(H) \end{bmatrix}$$

with (matrix) multiplication

$$\begin{bmatrix} \alpha & z \\ w^* & \beta \end{bmatrix} \times \begin{bmatrix} \alpha' & z' \\ w'^* & \beta' \end{bmatrix} = \begin{bmatrix} \alpha\alpha' + zw'^* & \alpha z' + z\beta' \\ w^*\alpha' + \beta w'^* & w^*z' + \beta\beta' \end{bmatrix},$$

and involution

$$\begin{bmatrix} \alpha & z \\ w^* & \beta \end{bmatrix}^* = \begin{bmatrix} \alpha^* & w \\ z^* & \beta^* \end{bmatrix}, \tag{2.1}$$

which is *-isomorphic to $\mathcal{A}_0(M)$ via the map

$$\mathcal{A}_0(M) \ni \begin{bmatrix} \ell(x,y) & z \\ \overline{w} & r(u,v) \end{bmatrix} \mapsto \begin{bmatrix} xy^* & z \\ w^* & u^*v \end{bmatrix} \in A_M. \tag{2.2}$$

We have the following companion result, which is fundamental in this paper, and whose proof is straightforward, and hence omitted.

Proposition 2.2. *Let $M \subset B(H,K)$ be an anti-TRO, that is, as a set, M is equal to a sub-TRO of $B(H,K)$, and it is considered as a C^*-ternary ring with the triple product $[xyz] := -xy^*z$. Then the standard embedding $\mathcal{A}_0(M)$ is *-isomorphic to the *-algebra*

$$\mathcal{B}_0 = \left\{ \begin{bmatrix} \sum_i x_i y_i^* & z \\ w^* & \sum_j u_j^* v_j \end{bmatrix} : x_i, y_i, u_j, v_j, z, w \in M \right\} \subset \begin{bmatrix} B(K) & B(H,K) \\ B(K,H) & B(H) \end{bmatrix}$$

with multiplication

$$\begin{bmatrix} \alpha & z \\ w^* & \beta \end{bmatrix} \cdot \begin{bmatrix} \alpha' & z' \\ w'^* & \beta' \end{bmatrix} = \begin{bmatrix} -\alpha\alpha' + zw'^* & -\alpha z' - z\beta' \\ -w^*\alpha' - \beta w'^* & w^*z' - \beta\beta' \end{bmatrix}, \tag{2.3}$$

and involution (same as (2.1))

$$\begin{bmatrix} \alpha & z \\ w^* & \beta \end{bmatrix}^* = \begin{bmatrix} \alpha^* & w \\ z^* & \beta^* \end{bmatrix}, \tag{2.4}$$

via the map (same as (2.2))

$$\mathcal{A}_0(M) \ni \begin{bmatrix} \ell(x,y) & z \\ \overline{w} & r(u,v) \end{bmatrix} \mapsto \begin{bmatrix} xy^* & z \\ w^* & u^*v \end{bmatrix} \in \mathcal{B}_0.$$

Definition 2.3. By an *anti-C^*-algebra* is meant a Banach algebra of the form $\mathcal{A}(M)$, for some anti-TRO M.

Example 2.4. In Proposition 2.2, if $M = \mathbb{C}$ with triple product $-x\overline{y}z$, then $\mathcal{A}_0(M)$ is equal to $M_2(\mathbb{C})$ as a linear space and is an associative *-algebra, with multiplication and involution given by (2.3) and (2.4) respectively. This anti-C*-algebra $(M_2(\mathbb{C}), \cdot)$ has a unit element

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

and like its counterpart $(M_2(\mathbb{C}), \times)$ ($\times$ denoting matrix multiplication), it has a trivial center and no nonzero proper two-sided ideals. An element

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if the "determinant" $ad + bc \neq 0$.

Remark 2.5. Since the anti-C*-algebra $(M_2(\mathbb{C}), \cdot)$ is central and simple, according to Wedderburn's theorem, it is isomorphic to the C*-algebra $(M_2(\mathbb{C}), \times)$. However, this isomorphism cannot be a *-isomorphism, since in that case it would contradict the last statement in Theorem 2.7 below.

The multiplication table for $(M_2(\mathbb{C}), \cdot)$ is

	E_{11}	E_{12}	E_{21}	E_{22}
E_{11}	$-E_{11}$	$-E_{12}$	0	0
E_{12}	0	0	E_{11}	$-E_{12}$
E_{21}	$-E_{21}$	E_{22}	0	0
E_{22}	0	0	$-E_{21}$	$-E_{22}$

and a Wedderburn isomorphism $\phi : (M_2(\mathbb{C}), \cdot) \to (M_2(\mathbb{C}), \times)$, given by

$$\phi(E_{ij}) = \sum_{p,q} a_{ijpq} E_{pq},$$

satisfies, for all p, s, i, j, ℓ, the 32 nonlinear equations with 16 unknowns a_{ijpq}

$$\sum_q a_{ijpq} a_{j\ell qs} = \epsilon(ij\ell) a_{i\ell ps},$$

where

$$\phi(E_{ij}) \times \phi(E_{j\ell}) = \phi(E_{ij} \cdot E_{j\ell}) = \epsilon(ij\ell) E_{i\ell}.$$

More generally, by Theorem 3.9, any finite dimensional anti-C*-algebra is semisimple and hence isomorphic, but not *-isomorphic to a C*-algebra. It follows that if M is an infinite direct sum of finite dimensional anti-TROs, then $\mathcal{A}(M)$ is an infinite dimensional anti-C*-algebra which is isomorphic, but not *-isomorphic, to a C*-algebra.

Recall from [15, Rem. 1.4] that a C*-ternary ring M is a right $R(M)^{op}$-Banach module. It was proved in [15, Prop. 2.3(iv)] that if M is a right $R(M)^{op}$-Hilbert module, then the normed standard embedding $\mathcal{A}(M)$ can be normed to be a C*-algebra. The following proposition is the converse of the latter result.

Proposition 2.6. *If M is a C*-ternary ring, then the normed standard embedding $\mathcal{A}(M)$ can be normed to be a C*-algebra if and only if M is a right $R(M)^{op}$-Hilbert module.*

Proof. As noted above, one direction has been proved in [15, Prop. 2.3(iv)]. Suppose then that $\mathcal{A}(M)$ can be normed to be a C*-algebra. Since M is an associative triple subsystem of the supposed C*-algebra $\mathcal{A}(M)$ it is therefore isomorphic as a C*-ternary ring to a TRO, say $V \subset B(H)$. By [15, Prop. 2.6 and Example 1.6], if $\phi : M \to V$ is the isomorphism, then the map

$$\mathcal{A}_0(M) \ni \begin{bmatrix} ([gh\cdot], [hg\cdot]) & f \\ \overline{g} & ([\cdot hk], [\cdot kh]) \end{bmatrix} \mapsto \begin{bmatrix} \phi(g)\phi(h)^* & \phi(f) \\ \phi(g)^* & \phi(h)^*\phi(k) \end{bmatrix} \in A_V$$

extends to a *-isomorphism of $\mathcal{A}(M)$ onto the linking algebra

$$A_V = \begin{bmatrix} \overline{VV^*} & V \\ V^* & \overline{V^*V} \end{bmatrix}$$

of V.

Let us denote the C*-algebra $\overline{V^*V}$ by $\mathcal{D}$. The TRO V is a right $\mathcal{D}$-Hilbert module with inner product $\langle u, v\rangle_{\mathcal{D}} = v^*u$. With γ denoting the *-isomorphism of $R(M)$ onto $\mathcal{D}$ ($\gamma : [\cdot hk], [\cdot kh]) \mapsto \phi(h)^*\phi(k))$, the map $\langle f, g\rangle_{R(M)} := \gamma^{-1}(\langle \phi(f), \phi(g)\rangle_{\mathcal{D}})$ is an $R(M)$-valued inner product making M into a right $R(M)^{op}$-Hilbert module. Indeed, it is obvious that $\langle f, f\rangle_{R(M)} \geq 0$, $\langle f, g\rangle^*_{R(M)} = \langle g, f\rangle_{R(M)}$, and

$$\|\langle f, f\rangle_{R(M)}\| = \|\langle \phi(f), \phi(f)\rangle_{\mathcal{D}}\| = \|\phi(f)\|^2 = \|f\|^2.$$

It remains to show that[2] $\langle f \cdot r(h, k), g\rangle_{R(M)} = \langle f, g\rangle_{R(M)} \circ r(h, k)$. First,

$$\langle f \cdot r(h, k), g\rangle_{R(M)} = \langle [fhk], g\rangle_{R(M)} = \gamma^{-1}\langle \phi[fhk], \phi(g)\rangle_{\mathcal{D}} = \gamma^{-1}(\phi(g)^*\phi[fhk]) = ([\cdot g[fhk]], [\cdot[fhk]g]),$$

and second, using [15, Lemma 1.1(iii)],

$$\begin{aligned}
\langle f, g\rangle_{R(M)} \circ r(h, k) &= \gamma^{-1}(\langle \phi(f), \phi(g)\rangle_{\mathcal{D}}) \circ ([\cdot hk], [\cdot kh]) \\
&= \gamma^{-1}(\phi(g)^*\phi(f)) \circ ([\cdot hk], [\cdot kh]) \\
&= ([\cdot gf], [\cdot fg]) \circ ([\cdot hk], [\cdot kh]) \\
&= ([\cdot hk], [\cdot kh])([\cdot gf], [\cdot fg]) \\
&= ([\cdot hk][\cdot gf], [\cdot fg][\cdot kh]) \\
&= ([[\cdot gf]hk], [[\cdot kh]fg],
\end{aligned}$$

as desired. □

In general, the normed standard embedding $\mathcal{A}(M)$ of a C*-ternary ring is a *-algebra which can be normed to be a C*-algebra if M is an R^{op}-Hilbert module by using a *-isomorphism π into $B(M \oplus R)$, with $M \oplus R$ considered as a Hilbert R^{op}-module ([15, Section 6]). If M is not a Hilbert R^{op}-module, then $M \oplus R$ is just a Banach space (under any convenient ℓ^p_2-norm). The proof that π is a homomorphism in this case is the same as the proof in the Hilbert module case in [15, pp. 34-35]. The proof given in [15, p. 36] for completeness works in this case also.

In the Hilbert module case in [15, Section 6], π was shown to be injective by using the fact that π was a self adjoint mapping, which however does not make sense if $B \oplus M$ is not a Hilbert module over R^{op}. However, a modification of that proof will now be given in the following theorem, which is the replacement for [15, Prop. 2.7], to show that in this case, π is injective.

Theorem 2.7. *The normed standard embedding $\mathcal{A}(M)$ of a C*-ternary ring M is the direct sum of a C*-algebra and a semisimple Banach algebra $\mathcal{B}$ with a continuous involution and a bounded approximate identity. The algebra $\mathcal{B}$ cannot be renormed to be a C*-algebra.*

Proof. By Zettl's decomposition ([15, Theorem 2.1]), $M = M_+ \oplus M_-$, where M_+ is isomorphic as a C*-ternary ring to a TRO, and M_- is isomorphic as a C*-ternary ring to an anti-TRO. By Remark 2.1 and [15, Prop. 2.6], $\mathcal{A}(M_+)$ is *-isomorphic to a C*-algebra. Moreover $\mathcal{A}(M) = \mathcal{A}(M_+) \oplus \mathcal{A}(M_-)$. It remains to show that $\mathcal{B} = \mathcal{A}(M_-)$ has the required properties.

Recall that the map $\pi : \mathcal{A}(M) \to B(M \oplus R)$ to the bounded operators on the Banach space $M \oplus R$, normed as above, say for definiteness,

$$\left\| \begin{bmatrix} f' \\ B' \end{bmatrix} \right\| = \left(\|f'\|^2 + \|B'\|^2\right)^{1/2},$$

is defined, for $a = \begin{bmatrix} A & f \\ \bar{g} & B \end{bmatrix} \in \mathcal{A}(M)$ by

$$\pi(a) \begin{bmatrix} f' \\ B' \end{bmatrix} = \begin{bmatrix} A & f \\ \bar{g} & B \end{bmatrix} \begin{bmatrix} f' \\ B' \end{bmatrix} = \begin{bmatrix} A \cdot f' + f \cdot B' \\ r(g, f') + B \circ B' \end{bmatrix}. \tag{2.5}$$

We have,

$$\|\pi(a)\| = \left\| \begin{bmatrix} A & f \\ \bar{g} & B \end{bmatrix} \right\| = \sup_{\|(f', B')\| \leq 1} \left\| \begin{bmatrix} A \cdot f' + f \cdot B' \\ r(g, f') + B \circ B' \end{bmatrix} \right\|,$$

[2] Recall that $\circ$ denotes the product in $R(M)^{op}$.

so that

$$\left\|\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}\right\| \geq \sup_{\|f'\|\leq 1} (\|A\cdot f'\|^2 + \|r(g,f')\|^2)^{1/2} \geq \sup_{\|f'\|\leq 1} \|A_1 f\| = \|A_1\| = \|A\|,$$

and

$$\left\|\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}\right\| \geq \sup_{\|f'\|\leq 1} (\|A\cdot f'\|^2 + \|r(g,f')\|^2)^{1/2} \geq \sup_{\|f'\|\leq 1} \|r(g,f')\| = \sup_{\|f'\|\leq 1}\sup_{\|f''\|\leq 1} \|[f''f'g]\| = \|g\|,$$

the last equality holding by applying [6, Lemma 2.3(iv)] to M_+ and M_-. Similarly,

$$\left\|\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}\right\| \geq \|B\| \text{ and } \left\|\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}\right\| \geq \|f\|.$$

Thus, if

$$\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \to 0, \text{ then } \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}^{\#} = \begin{bmatrix} \overline{A} & g \\ \overline{f} & \overline{B} \end{bmatrix} \to 0.$$

Let us now show that π is injective. For $a = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \in \mathcal{A}$, if $\pi(a) = 0$, then by (2.5)

$$A\cdot f' + f\cdot B' = 0 \text{ and } r(g,f') + B\circ B' = 0$$

for all $f' \in M, B' \in R$, and in particular,

$$A\cdot f' = 0 \text{ and } f\cdot B' = 0, \tag{2.6}$$

and

$$r(g,f') = 0 \text{ and } B\circ B' = 0. \tag{2.7}$$

From (2.6) with $B' = r(f,f)$, $[fff] = 0$ so $f = 0$. From (2.7), $\overline{B}B = 0$ and $r(g,g) = 0$, so $B = 0$ and $g = 0$. From (2.6) with $A = \ell(g,h)$, $[ghf'] = 0$ so that $A_1 = L(g,h) = 0$, and $L(h,g) = L(g,h)^* = 0$ so $A = (L(g,h), L(h,g)) = 0$. By the same argument, if $A = \sum_i \ell(g_i,h_i)$, then $A = 0$. Now suppose $A \in L$, let $\epsilon > 0$ and choose $A' = \sum_i \ell(g_i,h_i)$ with $\|A - A'\| < \epsilon$. Then $\|A'\cdot f'\| = \|(A - A')\cdot f'\| \leq \epsilon\|f'\|$, so that $\|A\| \leq \|A - A'\| + \|A'\| < 2\epsilon$, and $A = 0$.

Next, if $(A_\gamma)_{\gamma\in\Gamma}$ is an approximate identity for the C*-algebra $L(M)$, and $(B_\delta)_{\delta\in\Delta}$ is an approximate identity for the C*-algebra $R(M)$, then

$$\left(\begin{bmatrix} A_\gamma & 0 \\ 0 & B_\delta \end{bmatrix}\right)_{(\gamma,\delta)\in\Gamma\times\Delta}$$

is an approximate identity for $\mathcal{A}(M)$. To see this, one needs to prove that $A_\gamma\cdot f \to f$ and $B_\gamma\cdot g \to g$. For example, if $A_\gamma = (A^1_\gamma, A^2_\gamma)$ and[3] $f = [ggg]$, then $A_\gamma\cdot f = A^1_\gamma f = A^1_\gamma L(g,g)g \to L(gg)g = f$.

We now have that $\mathcal{A}(M_-)$ is a Banach algebra with a continuous involution. Since a C*-ternary ring M is embedded in its standard embedding $\mathcal{A}(M)$ as a sub-associative triple system, it is not possible for $\mathcal{A}(M_-)$ to support a norm making it a C*-algebra, since in that case M_- would be isomorphic as a C*-ternary ring to a sub-TRO of the supposed C*-algebra $\mathcal{A}(M_-)$, which violates the uniqueness of the Zettl decomposition.

For the proof of semisimplicity, see Theorem 3.9. □

We close this section by summarizing some results already mentioned which characterize TROs in the context of C*-ternary rings. The equivalence of (i) (or (ii)) and (v) in [14] answered a question posed by Zettl in [19, p. 136], namely, "characterizing the C*-ternary rings which yield $T = I$", equivalently, that are isomorphic to a TRO. Conditions (iii) and (iv) provide two new answers to Zettl's question.

Theorem 2.8. *Let M be a C*-ternary ring. The following are equivalent:*

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[3]Cube roots exist in C*-ternary rings. It suffices to know that cube roots exist in TROs and in anti-TROs. Since a TRO can be made into a JB*-triple by symmetrizing its triple product, TROs have cube roots ([8, p. 1135]). Let M be an anti-TRO, $M \subset B(H,K)$, with triple product $[xyz] = -xy^*z$, and if $a \in M$, then $-a \in M$ so $-a = bb^*b$ for some $b \in M$ and $a = (-b)(-b^*)(-b) = -bb^*b = [bbb]$.

(i): *M is isomorphic as a C*-ternary ring to a TRO.*
(ii): $M_- = 0$.
(iii): *M is a right $R(M)^{op}$-Hilbert module.*
(iv): *$\mathcal{A}(M)$ can be normed to be a C*-algebra.*
(v): *M is a JB*-triple under the triple product*

$$\{abc\} = \frac{[abc] + [bca]}{2}.$$

Proof.
(i)⇔ (v): : [14, p. 342]
(i)⇔ (ii): : Zettl's representation theorem ([15, Theorem 2.1])
(iii)⇒ (iv): : [15, Prop. 2.3 (iv)]
(iv)⇒ (iii): : Proposition 2.6
(i)⇒ (iv): : [15, Lemma 2.6, Example 1.6]
(iv)⇒ (i): : see the penultimate paragraph of the proof of Theorem 2.7.

□

3. Semisimplicity

In this section, following closely [12], we prove that an anti-C*-algebra is semisimple. Along the way we prove that TROs and anti-TROs are semisimple.

Definition 3.1. A *left ideal* (resp. *right ideal*) I in an associative triple system M is a linear subspace which satisfies $[MMI] \subset I$ (resp. $[IMM] \subset I$). An *ideal* is both a left ideal and a right ideal which satisfies $[MIM] \subset I$. (In a C*-ternary ring, a subspace which is both a left ideal and a right ideal automatically satisfies $[MIM] \subset I$.)

With M a C*-ternary ring, we denote by $\widetilde{\mathcal{A}}(M)$ the unitization of $\mathcal{A}(M)$:

$$\widetilde{\mathcal{A}}(M) = \mathcal{A}(M) \oplus \mathbb{C}\ \mathrm{Id}_{E(M)\oplus E(M)^{op}} = \begin{bmatrix} L(M) + \mathbb{C}E_1 & M \\ \overline{M} & R(M) + \mathbb{C}E_2 \end{bmatrix}, \tag{3.1}$$

where $E_1 = \mathrm{Id}_{E(M)}$ and $E_2 = \mathrm{Id}_{E(M)^{op}}$. The Peirce components of $\widetilde{\mathcal{A}}(M)$ relative to the idempotent E_1 are therefore precisely the entries in the matrix (3.1). With $\widetilde{A}$ denoting $\widetilde{\mathcal{A}}(M)$,

$$\begin{aligned} \widetilde{A} &= \widetilde{A}_{11} \oplus \widetilde{A}_{10} \oplus \widetilde{A}_{01} \oplus \widetilde{A}_{00} \\ &= E_1\widetilde{A}E_1 \oplus E_1\widetilde{A}E_2 \oplus E_2\widetilde{A}E_1 \oplus E_2\widetilde{A}E_2 \\ &= (L(M) + \mathbb{C}E_1) \oplus M \oplus \overline{M} \oplus (R(M) + \mathbb{C}E_2). \end{aligned}$$

For any closed ideal I in $\widetilde{A}$, we have (cf. [12, Lemma 7, p. 18])

$$I = \oplus_{i,j\in\{0,1\}}(\widetilde{A}_{ij} \cap I).$$

Thus, if I is a closed ideal in $\mathcal{A}(M) \subset \widetilde{\mathcal{A}}(M)$, then

$$I = (I \cap L(M)) \oplus (I \cap M) \oplus (I \cap \overline{M}) \oplus (I \cap R(M)), \tag{3.2}$$

and $I \cap M$ is a closed ideal in M (cf. [12, Lemma 4, p. 31]). Indeed, if $b \in I \cap M$ and $x, y \in M$, then

$$\begin{bmatrix} 0 & [xyb] \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}^{\#} \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \in I,$$

and similarly $[bxy] \in I \cap M$, so $I \cap M$ is an ideal in M.

Moreover, if $I \subset L(M) \oplus R(M)$, then $I = 0$. Indeed, since $I \cap M = I \cap \overline{M} = 0$, $I = (L(M) \cap I) \oplus (R(M) \cap I)$. Let $A = (A_1, A_2) \in L(M) \cap I$ and $f \in M$. Then

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A \cdot f \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A_1 f \\ 0 & 0 \end{bmatrix} \in I \cap M = 0, \tag{3.3}$$

so $A_1 = 0$ and $A_2 = A_1^* = 0$, and $A = 0$. Similarly $R(M) \cap I = 0$.

Recall that the Jacobson radical of an associative algebra A is the ideal consisting of the set of elements $x \in A$ which are quasi-invertible in every homotope A_u of A, that is, for every $u \in A$, there exists $y \in A$ (depending on x and u) such that $y - x = xuy = yux$.

Definition 3.2. The (Jacobson) *radical* Rad M of an associative triple system, such as a C*-ternary ring M, is the set of elements $x \in M$ which are quasi-invertible in every homotope M_u of M, that is, for every $u \in M$, there exists $y \in M$ such that

$$y - x = [yux] = [xuy].$$

A C*-ternary ring is said to be *semisimple* if its radical is 0.

Lemma 3.3. *(cf.* [12, Lemma 5, p. 32]*) For x and u in a C*-ternary ring M, the following are equivalent:*

- **(i):** *x is quasi-invertible in M_u, that is, there exists $y \in M$ such that $y - x = [yux] = [xuy]$.*
- **(ii):** $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ *is quasi-invertible in* $\mathcal{A}(M)_{\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}^\#}$*, that is, there exists $Y \in \mathcal{A}(M)$ such that*

$$Y - \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = Y \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}^\# \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}^\# Y.$$

Proof. Assume that (i) holds. Then just check that

$$\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}^\# \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}^\# \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}.$$

Conversely, assume (ii) holds. With $Y = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}$, we have

$$\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} - \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \overline{u} & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \overline{u} & 0 \end{bmatrix} \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix},$$

which reduces to

$$\begin{bmatrix} A & f - x \\ \overline{g} & B \end{bmatrix} = \begin{bmatrix} 0 & \ell(f,u) \cdot x \\ 0 & r(B \cdot \overline{u}, x) \end{bmatrix} = \begin{bmatrix} \ell(x, \overline{u} \cdot A) & x \cdot r(u, f) \\ 0 & 0 \end{bmatrix}.$$

Thus, $A = 0, g = 0, B = 0$ and $f - x = [fux] = [xuf]$. □

With M a C*-ternary ring, and since Rad $\mathcal{A}$ is an ideal, then by (3.2) (cf. [12, Theorem 5, p. 18]),

$$\text{Rad } \mathcal{A} = [(\text{Rad } \mathcal{A}) \cap L(M)] \oplus [(\text{Rad } \mathcal{A}) \cap M] \oplus ([\text{Rad } \mathcal{A}) \cap \overline{M}] \oplus [(\text{Rad } \mathcal{A}) \cap R(M)].$$

Moreover

$$(\text{Rad } \mathcal{A}) \cap L(M) = \text{Rad } L(M) \tag{3.4}$$

and

$$(\text{Rad } \mathcal{A}) \cap R(M) = \text{Rad } R(M). \tag{3.5}$$

To prove (3.4), let $X = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \in (\text{Rad } \mathcal{A}) \cap L(M)$. Then for all $U = \begin{bmatrix} A'' & f'' \\ \overline{g''} & B'' \end{bmatrix} \in \mathcal{A}$, there exists $Q = \begin{bmatrix} A' & f' \\ \overline{g'} & B' \end{bmatrix} \in \mathcal{A}$ with $Q - X = QUX = XUQ$, which after calculation yields

$$A' - A = A'A''A = AA''A',$$

so $A \in \text{Rad } L(M)$ and $(\text{Rad } \mathcal{A}) \cap L(M) \subset \text{Rad } L(M)$.

To show the reverse inclusion[4] we use the following two lemmas.

[4]Since $L(M)$ and $R(M)$ are C*-algebras, their radicals vanish; the proof given here for the reverse inclusion is therefore valid for general associative triple systems.

Lemma 3.4. *(cf.* [12, Lemma 4, p. 12] *"Symmetry principle") Given x, y in an associative algebra A, x is quasi-invertible in A_y if and only if y is quasi-invertible in A_x. That is,*

$$\exists u \in A, u - x = uyx = xyu \tag{3.6}$$

if and only if

$$\exists w \in A, w - y = wxy = xyw. \tag{3.7}$$

Proof. From (3.6), we have

$$uy - xy = uyxy = xyuy,$$

which means that xy is quasi-invertible in A. With $w = uy$

$$w - xy = wxy = xyw$$

and

$$yw - yxy = ywxy = yxyw$$

so that

$$(yw + y) - y = yw = ywxy + yxy = yxyw + yxy$$

and

$$(yw + y) - y = (yw + y)xy = yx(yw + y)$$

and therefor y is quasi-invertible in A_x. The converse follows by interchanging x and y. □

Lemma 3.5. *(cf.* [12, Lemma 5, p. 13] *"Shifting principle") Let φ and ψ be endomorphisms of an associative algebra A which satisfy, for all $x, y, z \in A$,*

$$\varphi(x)z\varphi(y) = \varphi(x\psi(z)y), \tag{3.8}$$

and

$$\psi(x)z\psi(y) = \psi(x\varphi(z)y). \tag{3.9}$$

Then for every $x \in A$, x is quasi-invertible in $A_{\psi(y)}$ if and only if $\varphi(x)$ is quasi-invertible in A_y, that is,

$$\exists u \in A, u - x = x\psi(y)u = u\psi(y)x, \tag{3.10}$$

if and only if

$$\exists w \in A, w - \varphi(x) = \varphi(x)yw = wy\varphi(x). \tag{3.11}$$

In each case, $\varphi(u) = w$.

Proof. Applying φ to (3.10) using (3.8) yields

$$\varphi(u) - \varphi(x) = \varphi(x)y\varphi(u) = \varphi(u)y\varphi(x),$$

so that $\varphi(x)$ is quasi-invertible in A_y with quasi inverse $\varphi(u)$, proving one direction as well as the last statement.

Conversely, assuming that $\varphi(x)$ is quasi-invertible in A_y, then by Lemma 3.4, y is quasi-invertible in $A_{\varphi(x)}$, so there exists $u \in A$ with

$$u - y = u\varphi(x)y = y\varphi(x)u.$$

Applying ψ using (3.9) yields

$$\psi(u) - \psi(y) = \psi(u)x\psi(y) = \psi(y)x\psi(u),$$

so that $\psi(y)$ is quasi-invertible in A_x, and by Lemma 3.4, x is quasi-invertible in $A_{\psi(y)}$. □

We can now complete the proof of (3.4). We shall apply Lemma 3.5 with $\varphi = \psi = E_1 \cdot E_1$ in the case that $x = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = E_1 x E_1 \in \text{Rad } L(M)$. Then x is quasi-invertible in $\mathcal{A}_{\psi(y)}$ for every $y \in \mathcal{A}$ and therefore $x = \varphi(x)$ is quasi-invertible in $\mathcal{A}_y$ for every $y \in \mathcal{A}$, so $x \in \text{Rad } (\mathcal{A})$. This proves (3.4), and (3.5) follows by a parallel argument.

We have proved part of the following theorem (cf. [12, Theorem 3, p. 32]).

Theorem 3.6. *If M is a C^*-ternary ring, and $\mathcal{A}(M)$ its normed standard embedding, then*

(i): *Rad* $\mathcal{A} = [Rad\ L(M)] \oplus [Rad\ M] \oplus [\overline{Rad\ M}] \oplus [(Rad\ R(M)]$
(ii): *Rad M is an ideal in M*
(iii): *$\mathcal{A}(M)$ is semisimple if and only if M is semisimple.*

Proof. To finish the proof of (i), it suffices to show that Rad M = [Rad $\mathcal{A}(M)] \cap M$ and Rad $\overline{M}$ = [Rad $\mathcal{A}(M)] \cap \overline{M}$. Suppose that $x \in$ [Rad $\mathcal{A}(M)] \cap M$, that is, $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ is quasi-invertible in $\mathcal{A}(M)_Y$ for every $Y = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \in \mathcal{A}(M)$. Thus, there exists $Q = \begin{bmatrix} A' & f' \\ \overline{g'} & B' \end{bmatrix} \in \mathcal{A}(M)$ such that

$$Q - \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} YQ = QY \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix},$$

which leads to $A' = 0, B' = 0, g' = 0$ and $f' - x = [xgf'] = [f'gx]$, and this means that $x \in$ Rad M, that is, [Rad $\mathcal{A}(M)] \cap M \subset$ Rad M.

Conversely, let $x \in$ Rad M, so that x is quasi-invertible in M_u for all $u \in M$. Then by Lemma 3.3, $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ is quasi-invertible in $\mathcal{A}_{\begin{bmatrix} 0 & 0 \\ \overline{u} & 0 \end{bmatrix}}$. Define $\varphi, \psi : \mathcal{A}(M) \to \mathcal{A}(M)$ by

$$\varphi\left(\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}\right) = E_1 \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} E_2 = \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}$$

and

$$\psi\left(\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}\right) = E_2 \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} E_1 = \begin{bmatrix} 0 & 0 \\ \overline{g} & 0 \end{bmatrix}.$$

Then by Lemma 3.5, $\varphi\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ is quasi-invertible in $\mathcal{A}(M)_{\psi(y)}$, that is, $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \in$ [Rad $\mathcal{A}(M)]\cap$ M, proving that Rad M = [Rad $\mathcal{A}(M)] \cap M$.

Since Rad $\mathcal{A}$ is self-adjoint (X is quasi-invertible in $\mathcal{A}_U$ for all $U \in \mathcal{A}$ if and only if $X^\#$ is quasi-invertible in $\mathcal{A}_{U^\#}$ for all $U^\# \in \mathcal{A}$), we have
$\begin{bmatrix} 0 & 0 \\ \overline{u} & 0 \end{bmatrix} \in$ (Rad $\mathcal{A}) \cap \overline{M}$ if and only if $\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \in [($Rad $\mathcal{A}) \cap \overline{M}]^\# = ($Rad $\mathcal{A}) \cap M =$ Rad M. Therefore $\overline{\text{Rad } M} = ($Rad $\mathcal{A}) \cap \overline{M}$, completing the proof of (i), and also proving (ii) as in (3.2).

Finally, if Rad $\mathcal{A} = 0$, obviously, Rad $M = 0$. Conversely, if Rad $M = 0$, then Rad $\mathcal{A} \subset L(M) \oplus R(M)$ so as in (3.3), Rad $\mathcal{A} = 0$. ☐

Corollary 3.7. *If M is a TRO, then M is semisimple.*

Proposition 3.8. *Let $M \subset B(H, K)$ be an anti-TRO, that is, as a linear space, M is equal to a sub-TRO M' of $B(H, K)$, and it is considered as a C^*-ternary ring with the triple product $[xyz] := -xy^*z$. Then Rad M = Rad M'. Thus, an anti-TRO is semisimple.*

Proof. We note first that ideals in M are also ideals in M', and conversely. According to [13, Theorem 9], for any associative triple system N of the second kind, with ternary product denoted $[xyz]$,

(3.12) $\qquad$ Rad $N = \{x \in N :$ the principal ideal $[xNN]$ or $[NNx]$ is quasi-regular in $N\}$.

Quasi-regular for the right ideal $[xNN]$, which is equivalent to quasi-regularity for left ideals [13, p. 32], means that

$$N = \{u - [vyu] : u, y \in N, v \in [xNN]\}, \tag{3.13}$$

so in our case, we need to prove for the anti-TRO M, that

$$M = \{u + vy^*u : u, y \in M, v \in xM^*M\}.$$

By (3.12), for the TRO M',

$$\text{Rad } M' = \{x \in M' : \text{ the principal ideal } xM'^*M' \text{ or } M'M'^*x \text{ is quasi-regular in } M'\},$$

that is,

$$M' = \{u - vy^*u : u, y \in M', v \in xM'^*M'\}.$$

In addition to ideals of M and M' coinciding, quasi-regularity is preserved, that is,

$$\{u + vy^*u : u, y \in M, v \in xM^*M\} = \{u - v(-y)^*u : u, -y \in M, v \in xM^*M\} = M' = M,$$

proving (3.13) with $N = M'$. Hence Rad $M =$ Rad M'. □

Theorem 3.9. *The Banach algebra $\mathcal{B}$ in Theorem 2.7 is semisimple. Hence, all C*-ternary rings and all anti-C*-algebras are semisimple.*

Proof. This follows from Theorem 3.6(iii) and Proposition 3.8. □

4. Ideals

There is a one to one correspondence between closed ideals in a TRO M and closed ideals in the C*-algebra $R(M)$. This was proved directly (that is, not using [17]) in [2, Prop. 3.8]. This result was extended to C*-ternary rings in [1, Prop. 4.2], by using the TRO $M_+ \oplus M_-^{op}$ and appealing to [17, Theorem 3.22]. It can also be proved by using the TRO $M_+ \oplus M_-^{op}$ and appealing to [2, Prop. 3.8] (in general, if $(M, [\cdot,\cdot,\cdot])$ is a C*-ternary ring, then M^{op} denotes the C*-ternary ring $(M, -[\cdot,\cdot,\cdot])$).

We now consider the corresponding property in the context of C*-ternary rings. We begin by modifying some earlier notation. Recall that if M is a C*-ternary ring, $\ell(f,g)$, for $f, g \in M$, is the element of $B(M \oplus M)$ defined by $\ell(f,g)(x,y) = ([fgx],[gfy])$. To emphasize the dependence on M we denote $\ell(f,g)$ by $\ell_M(f,g)$. Thus

$$L(M) := \overline{sp}\{\ell_M(f,g) : f, g \in M\} \subset B(M \oplus M).$$

Let I be a closed ideal in the C*-ternary ring M. Since I is also a C*-ternary ring,

$$L(I) := \overline{sp}\{\ell_I(f,g) : f, g \in I\} \subset B(I \oplus I).$$

We now define

$$\tilde{L}(I) = \overline{sp}\{\ell_M(f,g) : f, g \in I\} \subset L(M) \subset B(M \oplus M).$$

Note that

$$\ell_I(f,g) = \ell_M(f,g)|_{I \oplus I}.$$

Similarly, recall that $r(f,g)$, for $f, g \in M$, is the element of $B(M \oplus M)$ defined by $r(f,g)(x,y) = ([xfg],[ygf)$. To emphasize the dependence on M we denote $r(f,g)$ by $r_M(f,g)$. Thus

$$R(M) := \overline{sp}\{r_M(f,g) : f, g \in M\} \subset B(M \oplus M).$$

For a close ideal I in M, we have

$$R(I) := \overline{sp}\{r_I(f,g) : f, g \in I\} \subset B(I \oplus I),$$

and we define

$$\tilde{R}(I) = \overline{sp}\{r_M(f,g) : f, g \in I\} \subset R(M) \subset B(M \oplus M).$$

Note that

$$r_I(f,g) = r_M(f,g)|_{I \oplus I}.$$

The proof of the following proposition is a straightforward application of the definitions.

Proposition 4.1. *Let I be a closed ideal in the C*-ternary ring M.*

- **(a):** *$\tilde{L}(I)$ is a closed two-sided ideal in the C*-algebra $L(M)$, and the map ϕ defined as $\phi(\ell_M(f,g)) = \ell_I(f,g)$, for $f, g \in I$, extends to a contractive *-homomorphism of $\tilde{L}(I)$ onto $L(I)$.*
- **(b):** *$\tilde{R}(I)$ is a closed two-sided ideal in the C*-algebra $L(M)$, and the map ψ defined as $\psi(r_M(f,g)) = r_I(f,g)$, for $f, g \in I$, extends to a contractive *-homomorphism of $\tilde{R}(I)$ onto $R(I)$.*

We next define

$$\tilde{\mathcal{A}}(I) = \begin{bmatrix} \tilde{L}(I) & I \\ \overline{I} & \tilde{R}(I) \end{bmatrix} \subset \mathcal{A}(M) = \begin{bmatrix} L(M) & M \\ \overline{M} & R(M) \end{bmatrix}.$$

The proof of the following proposition is also a straightforward application of the definitions. It is included for the convenience of the reader and to motivate one of the questions which follows it.

Proposition 4.2. *Let I be a closed ideal in a C*-ternary ring M. Then $\tilde{\mathcal{A}}(I)$ is a closed self-adjoint ideal in $\mathcal{A}(M)$. The map $I \mapsto \tilde{\mathcal{A}}(I)$ is injective from closed ideals of M to closed self-adjoint two-sided ideals of $\mathcal{A}(M)$.*

Proof. We show for example that $\mathcal{A}(I)$ is a right ideal in $\mathcal{A}(M)$. For

$$\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \in \tilde{\mathcal{A}}(I) \text{ and } \begin{bmatrix} A' & f' \\ \overline{g'} & B' \end{bmatrix} \in \mathcal{A}(M)$$

we have

$$\begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \begin{bmatrix} A' & f' \\ \overline{g'} & B' \end{bmatrix} = \begin{bmatrix} AA' + \ell(f,g') & A \cdot f' + f \cdot B' \\ \overline{g} \cdot A' + B \cdot \overline{g'} & r(g,f') + B \circ B' \end{bmatrix}.$$

and it is required to show that

$$AA' + \ell(f,g') \in \tilde{L}(I), \tag{4.1}$$

$$A \cdot f' + f \cdot B' \in I \tag{4.2}$$

$$\overline{g} \cdot A' + B \cdot \overline{g'} \in \overline{I} \tag{4.3}$$

$$r(g,f') + B \circ B' \in \tilde{R}(I). \tag{4.4}$$

To prove (4.1) and (4.2), we may assume that $A = \ell_M(g,h)$ for $g,h \in I$, $A' = \ell_M(g',h')$ for $g',h' \in M$, and $B' = r_M(g'',h'')$ for $g'',h'' \in M$. By [15, Lemma 1.1(ii)],

$$AA' = \ell_M(g,h)\ell_M(g',h') = \ell_M(g,[h'g'h]) \in \tilde{L}(I).$$

Write $f = [ggg]$ for some $g \in I$ (see footnote 3). Then $\ell_M(f,g') = \ell_M(g,[g'gg]) \in \tilde{L}(I)$, proving (4.1). Since $A \cdot f' = [ghf'] \in I$ and $f \cdot B' = [fg''h''] \in I$, this proves (4.2). (4.3) and (4.4) are proved similarly. □

The following proposition was stated in [1, Prop. 4.5] as a special case of [17, Prop. 3.25]. We present here a direct proof based on the Zettl representation.

Proposition 4.3. *If M is a C*-ternary ring, and J is a closed ideal in M, then the quotient M/J is a C*-ternary ring.*

Proof. As in the proof of [15, Lemma 2.4], the decomposition $M = M_+ \oplus M_-$ is obtained by

$$M_+ = \{f \in M : \alpha(f,f) \geq 0\}, \quad M_- = \{f \in M : \alpha(f,f) \leq 0\},$$

where α is defined[5] by $\alpha(f,g) = r(g,f)$ for $f,g \in M$.

It is now easy to check that with $\alpha' = \alpha_{|J \times J}$, and $\mathfrak{A}'$ the closed span of $\alpha(J,J)$, then the C*-ternary ring J, together with α' and $\mathfrak{A}'$, satisfy the conditions stated in [15, Rem. 2.2]. It follows that the Zettl decomposition of J is $J = J_+ \oplus J_- = (J \cap M_+) \oplus (J \cap M_-)$ and that algebraically,

$$M/J = (M_+/J_+) \oplus (M_-/J_-)\,.$$

Finally, since M_+ is isomorphic as a C*-ternary ring to a TRO, and J_+ is a closed ideal in M_+, it follows from [4, Prop. 2.2], that M_+/J_+ is isomorphic to a TRO. As M_- is isomorphic to an anti-TRO, the same argument shows that M_-/J_- is isomorphic to an anti-TRO, and therefore M/J is a C*-ternary ring. □

Some questions

Question 1. Which C*-algebras can appear as the linking algebra of a TRO? Which semisimple Banach algebras with approximate identities and with continuous involution can appear as anti-C*-algebras?

Question 2. In what sense is the decomposition in Theorem 2.7 unique? In particular, if M and N are C*-ternary rings, with *-isomorphic normed standard embeddings, does it follow that M and N are isomorphic?

[5] In [15, Lemma 2.4], α was defined incorrectly as $\alpha(f,g) = r(f,g)$.

Question 3. If M is a C*-ternary ring, is the map $I \mapsto \tilde{\mathcal{A}}(I)$ in Proposition 4.2 from closed ideals of M to closed self-adjoint two-sided ideals of $\mathcal{A}(M)$ surjective? In the special case that M is a TRO (and hence $\mathcal{A}(M)$ is a C*-algebra), this has been proved in [18, Prop. 2.7].

Question 4. If M is a C*-ternary ring, then its bidual M^{**} is a C*-ternary ring ([9, Theorem 2]). What is the relation between $L(M)^{**}$ and $L(M^{**})$, between $R(M)^{**}$ and $R(M^{**})$, between $\mathcal{A}(M)^{**}$ and $\mathcal{A}(M^{**})$?

5. Ternary Operator Categories—Revisited

5.1. **Adjustments to** [15, Section 4]**.** In what follows, we shall use some notation and some results from [15], making precise references to [15] when necessary. We first modify, as suggested in [15], the definition of linking category. Recall that the morphism sets $(X, Y)_{\mathcal{C}}$ in a linear ternary category are associative triple systems.

Definition 5.1. (Modification of [15, Def. 3.8]) Given a linear ternary category $\mathcal{C}$, the *linking category* $A_{\mathcal{C}}$ of $\mathcal{C}$ is as follows. The objects of the category $A_{\mathcal{C}}$ are the same as the objects of $\mathcal{C}$. The morphism set $\mathrm{Hom}(X, Y)$ $(= (X, Y)_{A_{\mathcal{C}}})$ is defined to be $\mathcal{A}(X, Y)$ $(= \mathcal{A}((X, Y)_{\mathcal{C}})$, with composition as follows. If $a \in \mathrm{Hom}(X, Y)$ and $b \in \mathrm{Hom}(Y, Z)$, then $b \circ a$ must be 0 unless $X = Y = Z$, in which case $b \circ a$ is defined to be the product ab in $\mathcal{A}(X, X)$.

Remark 5.2. The category $A_{\mathcal{C}}$ can be considered as a ternary category ([15, Definition 3.1]) under the composition $[abc] = ab^{\#}c$ and, by [15, Lemma 1.3], we obtain an injective linear ternary functor F from $\mathcal{C}$ to $A_{\mathcal{C}}$ by associating the object X of $\mathcal{C}$ to the object X of $A_{\mathcal{C}}$ and the morphism $f \in (X, Y)$ to the morphism $\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} \in \mathcal{A}(X, Y)$.

Recall that the morphism sets in a T*-category ([15, Definition 4.6]) are C*-ternary rings ([15, Section 2]), so they satisfy Zettl's representation theorem [15, Theorem 2.1].

Definition 5.3. Given a T*-category $\mathcal{C}$, the *positive (resp. negative) linking category* $A_{\mathcal{C}}^{\pm}$ of $\mathcal{C}$ is as follows. The objects of the category $A_{\mathcal{C}}^{\pm}$ are the same as the objects of $\mathcal{C}$. The morphism set $\mathrm{Hom}(X, Y)$ is defined to be $\mathrm{Hom}(X, Y) = \mathcal{A}((X, Y)_{\pm})$, with composition as follows. If $a \in \mathrm{Hom}(X, Y)$ and $b \in \mathrm{Hom}(Y, Z)$, then $b \circ a$ must be 0 unless $X = Y = Z$, in which case $b \circ a$ is defined to be the product ab in $\mathcal{A}((X, X)_{\pm})$.

Thus the linking categories $A_{\mathcal{C}}^{\pm}$ of the T*-category $\mathcal{C}$ are subcategories of the linking category $A_{\mathcal{C}}$ of the linear ternary category $\mathcal{C}$, and $A_{\mathcal{C}} = A_{\mathcal{C}}^{+} \oplus A_{\mathcal{C}}^{-}$, by [15, Remark 4.13(ii)].

Theorem 5.4. *(Replacement for* [15, Theorem 4.11]*) If $\mathcal{C}$ is a T*-category then $A_{\mathcal{C}}^{+}$ is a C*-category (*[15, Definition 4.1]*) and there is a faithful T*-functor F from $\mathcal{C}_{+}$ to $A_{\mathcal{C}}^{+}$, the latter considered as a T*-category.*

Proof. It is clear that $A_{\mathcal{C}}^{+}$, as defined in Definition 5.3, is a linear non-unital category which, when considered as a ternary category, satisfies (ii), (iii) and (vi) in [15, Def. 4.1]. Items (i), (iv), and (v) of that definition are tantamount to the morphism sets of $A_{\mathcal{C}}^{+}$, namely, $\mathcal{A}((X, Y)_{+})$, being normed as C*-algebras. This fact is immediate from Theorem 2.7, and the faithful functor is the restriction of the functor defined in Remark 5.2. □

Let $\rho = (\rho_0, \{\rho_{X,Y}\})$ be a linear ternary functor from a linear ternary category $\mathcal{C}$ to a a linear ternary category $\mathcal{D}$. We write $\rho = (\rho_0, \{\rho_{X,Y} : X, Y \text{ objects of } \mathcal{C}\})$, where ρ_0 maps objects of $\mathcal{C}$ to objects of $\mathcal{D}$, and $\rho_{X,Y}$ is a linear transformation from $(X, Y)_{\mathcal{C}}$ to $(\rho_0(X), \rho_0(Y))_{\mathcal{D}}$ satisfying

$$\rho_{X,W}(h \circ g^* \circ f) = \rho_{Z,W} \circ \rho_{Z,Y}(g)^* \circ \rho_{X,Y}(f),$$

for $f \in (X, Y)_{\mathcal{C}},\ g \in (Z, Y)_{\mathcal{C}},$ and $h \in (Z, W)_{\mathcal{C}}$.

If $\mathcal{C}$ is a T*-category, T*-subcategories $\mathcal{C}_{\pm}$ are defined as follows. By Zettl's representation theorem, we have $(X, Y) = (X, Y)_{+} \oplus (X, Y)_{-}$ for each pair of objects X, Y of $\mathcal{C}$. The objects of $\mathcal{C}_{\pm}$ are the same as the objects of $\mathcal{C}$, and for such objects X, Y,

$$(X, Y)_{\mathcal{C}_{\pm}} := (X, Y)_{\pm}.$$

It is clear that $\mathcal{C}$ is isomorphic to $\mathcal{C}_+ \oplus \mathcal{C}_-$ and that $A_{\mathcal{C}}$ is isomorphic to $\mathcal{A}_{C_+} \oplus \mathcal{A}_{C_-}$ (cf. [15, Remark 4.13(ii)]).

Remark 5.5. (Replacement for [15, Remark 4.12]) If ρ_0 is injective, then there is a linear functor $\widehat{\rho}$ from $A_{\mathcal{C}}$ to $A_{\mathcal{D}}$ which extends ρ. In particular, if $\mathcal{C}$ and $\mathcal{D}$ are T*-categories, then every T*-functor from $\mathcal{C}_+$ to $\mathcal{D}_+$, which is injective on objects, extends to a C*-functor from $A^+_{\mathcal{C}}$ to $A^+_{\mathcal{D}}$.

Proof. Recall that $(X,Y)_{\mathcal{C}}$ is an associative triple system, and $\rho_{X,Y}$ is a homomorphism of $(X,Y)_{\mathcal{C}}$ to $(\rho_0(X),\rho_0(Y))_{\mathcal{D}}$, so by [15, Lemma 2.6], it extends to a *-homomorphism $\widehat{\rho}_{X,Y} = \mathcal{A}(\rho_{X,Y})$ from $(X,Y)_{A_{\mathcal{C}}}$ to $(\rho_0(X),\rho_0(Y))_{A_{\mathcal{D}}}$.

Thus $\widehat{\rho} = (\widehat{\rho}_0, \{\widehat{\rho}_{X,Y} : X, Y \text{ objects of } A_{\mathcal{C}}\})$, where $\widehat{\rho}_0(X) = \rho_0(X)$, is the desired linear functor from $A_{\mathcal{C}}$ to $A_{\mathcal{D}}$ whose restriction to $\mathcal{A}^+_{\mathcal{C}}$ is a C*-functor to $\mathcal{A}^+_{\mathcal{D}}$. □

Theorem 5.6. *(Replacement for* [15, Theorem 4.14]*) Let $\mathcal{C}$ be a T*-category. Then there is a faithful T*-functor from $\mathcal{C}_+$ to the T*-category $\mathcal{H}$ of Hilbert spaces and continuous linear maps (*[15, Example 4.2, Remark 4.8]*).*

Proof. By [15, Theorem 4.5], there is a faithful C*-functor G_+ from $A_{\mathcal{C}_+}$ to $\mathcal{H}$. With $A_{\mathcal{C}_+}$ considered as a T*-category, we have that G_+ is a T*-functor from $A_{\mathcal{C}_+}$ to $\mathcal{H}$. By Theorem 5.4, there is a faithful T*-functor F_+ from $\mathcal{C}_+$ to $A_{\mathcal{C}_+}$, and it suffices to consider $H = G_+ \circ F_+$. □

Let V be a C*-ternary ring. By Theorem 2.7, V is the off-diagonal corner of a Banach algebra with continuous involution (a C*-algebra if V is a TRO), $\mathcal{A}(V)$, where

$$\mathcal{A}(V) = \begin{bmatrix} L & V \\ \overline{V} & R \end{bmatrix},$$

and $L = L(V)$ and $R = R(V)$ are C*-algebras. Consider

$$\tilde{A}(V) = \begin{bmatrix} M(L) & V \\ \overline{V} & M(R) \end{bmatrix}, \tag{5.1}$$

where $M(L)$ and $M(R)$ are the multiplier algebras of L and of R. Zettl has shown in [19, Prop. 4.9] that if V is a W*-ternary ring ([15, Section 2]), then $M(R)$ and consequently $M(L)$ are W*-algebras.

Recall that a T*-category is a *TW*-category* if each morphism set is a dual Banach space, and that for objects X, Y in a TW*-category, (X,Y) is a W*-ternary ring.

Definition 5.7. (Replacement for [15, Def. 4.22]) Given a TW*-category $\mathcal{C}$ ([15, Definition 4.6]), the *linking W*-category* $\tilde{A}^+_{\mathcal{C}}$ of $\mathcal{C}$ is as follows. The objects of the category $\tilde{A}^+_{\mathcal{C}}$ are the same as the objects of $\mathcal{C}$. The morphism set $\mathrm{Hom}(X,Y)$ is defined to be $\tilde{\mathcal{A}}((X,Y)_+)$, as in (5.1) with $V = (X,Y)_+$, and with composition as follows. If $a \in \mathrm{Hom}(X,Y)$ and $b \in \mathrm{Hom}(Y,Z)$, then $b \circ a$ must be 0 unless $X = Y = Z$, in which case $b \circ a$ is defined to be the product ab in $\tilde{\mathcal{A}}((X,X)_+)$.

Theorem 5.8. *(Replacement for* [15, Theorem 4.23]*) If $\mathcal{C}$ is a TW*-category then $\tilde{A}^+_{\mathcal{C}}$ is a W*-category (C*-category with morphism sets being dual spaces) and there is a faithful TW*-functor F from $\mathcal{C}_+$ to $\tilde{A}^+_{\mathcal{C}}$, the latter considered as a TW*-category.*

Proof. It is clear that $\tilde{A}^+_{\mathcal{C}}$, as defined in Definition 5.7, is a linear non-unital category which, when considered as a ternary category, satisfies (ii), (iii) and (vi) in [15, Def. 4.6]. Items (i), (iv), and (v) of that definition are tantamount to the morphism sets of $\tilde{A}^+_{\mathcal{C}}$, namely, $\mathcal{A}((X,Y)_+)$, being normed as W*-algebras. This fact is immediate from [15, Prop. 4.21]. □

Theorem 5.9. *(Replacement for* [15, Theorem 4.25]*) Let $\mathcal{C}$ be a TW*-category. Then there is a faithful TW*-functor from $\mathcal{C}_+$ to the TW*-category $\mathcal{H}$ of Hilbert spaces and continuous linear maps.*

Proof. By [5, Prop. 2.13], there is a faithful W*-functor F from $\tilde{A}^+_{\mathcal{C}}$ to $\mathcal{H}$. With $\tilde{A}^+_{\mathcal{C}}$ considered as a TW*-category, we have that G is a TW*-functor from $\tilde{A}^+_{\mathcal{C}}$ to $\mathcal{H}$. By Theorem 5.8, there is a faithful TW*-functor F from $\mathcal{C}^+$ to $\tilde{A}^+_{\mathcal{C}}$, and it suffices to consider $H = G \circ F$. □

5.2. **Two completed proofs for** [15]**.**

Remark 5.10. (Completion of the proof of [15, Rem. 4.10]) In the proof of [15, Rem. 4.10], only items (i)-(iv) of the five conditions in [15, Def. 4.6] were proved. The remaining item (v) (as well as item (iv)) follows immediately from Proposition 4.3.

In the proof of [15, Prop. 5.13], it was stated incorrectly that C*-ternary rings are JB*-triples, and therefore, since JB*-triples satisfy Pelczynski's property V [3], so do C*-ternary rings. However, as pointed out in Theorem 2.8, a C*-ternary ring $M = M_+ \oplus M_-$ is (isomorphic to) a JB*-triple if and only if $M_- = 0$. Thus a C*-ternary ring is a JB*-triple if and only if it is isomorphic to a TRO. Using this fact, we can prove the following proposition, which completes the proof of [15, Prop. 5.13].

Proposition 5.11. *A C*-ternary ring $(M, [\cdot,\cdot,\cdot])$ satisfies Pelczynski's property V.*

Proof. According to the theorem of Zettl [15, Theorem 2.1], there is a bounded operator T on M such that $(M, T\circ[\cdot,\cdot,\cdot])$ is isomorphic to a TRO. Therefore $(M, T\circ[\cdot,\cdot,\cdot])$ satisfies Pelczynski's property V. Since $(M, [\cdot,\cdot,\cdot])$ and $(M, T\circ[\cdot,\cdot,\cdot])$ are identical as Banach spaces, the proposition is proved. □

Appendix. Standard Embedding

Let M be an associative triple system with triple product denoted by $[hgf]$. Let

$$E(M) = \text{End}\,(M) \oplus [\overline{\text{End}\,(M)}]^{op},$$

where the notation $\overline{V}$ for a complex vector space means that the scalar multiplication in $\overline{V}$ is $(\lambda, v) \in \mathbb{C}\times V \mapsto \lambda\circ v = \overline{\lambda}v$. We shall denote the products in $E(M)^{op}$ and in $[\text{End}\,(M)]^{op}$ by $X\circ Y := YX$.

Involutions are defined on $E(M)$ by

$$A = (A_1, A_2) \mapsto \overline{A} = \overline{(A_1, A_2)} = (A_2, A_1),$$

and on $E(M)^{op}$ by

$$B = (B_1, B_2) \mapsto \overline{B} = \overline{(B_1, B_2)} = (B_2, B_1).$$

For $g, h \in M$, define

$$L(g,h) = [gh\cdot], R(g,h) = [\cdot hg],$$
$$\ell(g,h) = (L(g,h), L(h,g)) = ([gh\cdot],[hg\cdot]) \in E(M),$$
$$r(g,h) = (R(h,g), R(g,h)) = ([\cdot gh],[\cdot hg]) \in E(M)^{op},$$
$$L_0 = L_0(M) = \text{span}\,\{\ell(g,h) : g,h \in M\} \subset E(M),$$
$$R_0 = R_0(M) = \text{span}\,\{r(g,h) : g,h \in M\} \subset E(M)^{op}.$$

Let $A = (A_1, A_2) \in E(M)$, $B = (B_1, B_2) \in E(M)^{op}$, and $f \in M$. Let $\overline{M}$ denote the vector space M with the element f denoted by $\overline{f}$ and with scalar multiplication defined by $(\lambda, \overline{f}) \mapsto \lambda\circ\overline{f} = \overline{\overline{\lambda}f}$. Then

- **(i):** M is a left $E(M)$-module via $(A, f) \mapsto A\cdot f = A_1 f$, a right $E(M)^{op}$-module via $(f, B) \mapsto f\cdot B = B_1 f$, and an (L_0, R_0)-bimodule.
- **(ii):** $\overline{M}$ is a left $E(M)^{op}$-module via $(B, \overline{f}) \mapsto B\cdot\overline{f} = \overline{B_2 f}$, a right $E(M)$-module via $(\overline{f}, A) \mapsto \overline{f}\cdot A = \overline{A_2 f}$, and an (R_0, L_0)-bimodule.

Given an associative triple system M, let

$$\mathcal{A}_0 = \mathcal{A}_0(M) = L_0(M) \oplus M \oplus \overline{M} \oplus R_0(M)$$

and write the elements a of $\mathcal{A}_0$ as matrices

$$a = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix} \in \begin{bmatrix} L_0(M) & M \\ \overline{M} & R_0(M) \end{bmatrix}.$$

Define multiplication and involution in $\mathcal{A}_0$ by

$$aa' = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}\begin{bmatrix} A' & f' \\ \overline{g'} & B' \end{bmatrix} = \begin{bmatrix} AA' + \ell(f, g') & A\cdot f' + f\cdot B' \\ \overline{g}\cdot A' + B\cdot\overline{g'} & r(g, f') + B\circ B' \end{bmatrix}$$

and

$$a^\sharp = \begin{bmatrix} A & f \\ \overline{g} & B \end{bmatrix}^\# = \begin{bmatrix} \overline{A} & g \\ \overline{f} & \overline{B} \end{bmatrix}.$$

Then $\mathcal{A}_0(M)$ is an associative *-algebra and for $f, g, h \in M$,

$$\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & g \\ 0 & 0 \end{bmatrix}^\# \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & [fgh] \\ 0 & 0 \end{bmatrix}.$$

The map $f \mapsto \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}$ is a triple isomorphism of M into $\mathcal{A}_0(M)$, the latter considered as an associative triple system with triple product $ab^\# c$, for $a, b, c \in \mathcal{A}_0(M)$, We refer to $\mathcal{A}_0(M)$ as the *standard embedding* of M. If the associative triple system M is a normed space, and $\|[hgf]\| \le \|f\|\|g\|\|h\|$, then the *normed standard embedding* of M, denoted by $\mathcal{A}(M)$, is defined in the same way but with $R_0(M)$ and $L_0(M)$ replaced by their closures $L(M)$ and $R(M)$ in $B(M \oplus M)$.

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