Quantum Information Theory
Bernie Russo

following

Michael Keyl
Fundamentals of quantum information theory
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Contents

1 January 3, 2007—Keyl, Section 2—Basic Concepts

1.1 Composite Systems and Entangled States (Keyl 2.2)

The tensor product $H \otimes K$ is defined to be the span of $\{\psi_1 \otimes \psi_2 : \psi_1 \in H, \psi_2 \in K\}$ where $\psi_1 \otimes \psi_2$ is the bilinear form defined by[1]

$$\psi_1 \otimes \psi_2(\phi_1, \phi_2) = (\psi_1|\phi_1)(\psi_2|\phi_2) \quad , \quad (\phi, \phi_2) \in H \times K.$$

The inner product in $H \otimes K$ is given by

$$(\psi_1 \otimes \psi_2|\eta_1 \otimes \eta_2) = (\psi_1|\eta_1)(\psi_2|\eta_2)$$

[1] Note that, following the physicists, inner products will be linear in the second variable

The tensor product $B(H) \otimes B(K)$ is defined[2] to be the span of $\{A_1 \otimes A_2 : A_1 \in B(H), A_2 \in B(K)\}$ where $A_1 \otimes A_2$ is the operator defined by

$$A_1 \otimes A_2(\psi_1 \otimes \psi_2) = A_1\psi_1 \otimes A_2\psi_2.$$

A *partial trace* of an operator $\rho \in B(H) \otimes B(K)$ is the operator $\mathrm{tr}_K(\rho) \in B(H)$ defined by[3]

$$\mathrm{tr}(\mathrm{tr}_K(\rho)A) = \mathrm{tr}(\rho \cdot (A \otimes 1)) \quad , \quad A \in B(H).$$

Symmetrically, another *partial trace* of an operator $\rho \in B(H) \otimes B(K)$ is the operator $\mathrm{tr}_H(\rho) \in B(K)$ defined by

$$\mathrm{tr}(\mathrm{tr}_H(\rho)B) = \mathrm{tr}(\rho \cdot (1 \otimes B)) \quad , \quad B \in B(K).$$

For example, if $\rho = B_1 \otimes B_2$, then $\mathrm{tr}_K(\rho) = \mathrm{tr}(B_2)B_1$. (Proof: $\rho \cdot (A \otimes 1) = B_1 A \otimes B_2$ and $\mathrm{tr}(\rho \cdot (A \otimes 1) = \mathrm{tr}(B_1 A)\mathrm{tr}(B_2) = \mathrm{tr}([\mathrm{tr}(B_2)B_1]A)$.)

Note that $\mathrm{tr}_K(\cdot)$ is a positive linear operator from $B(H) \otimes B(K)$ to $B(H)$.

Proposition 1.1 (Proposition 2.2, page 445) *For each element Ψ of the two-fold tensor product $H \otimes K$, there are orthonormal systems $\{\phi_j : j = 1, \ldots n\}$ and $\{\psi_k : k = 1, \ldots n\}$ (not necessarily bases, i.e. n can be smaller than $\dim H$ and $\dim K$) for H and K respectively, and non-negative numbers λ_j, such that $\Psi = \sum_j \sqrt{\lambda_j}\phi_j \otimes \psi_j$ holds. The ϕ_j and the ψ_k are uniquely determined by Ψ, and the expansion is called the Schmidt decomposition and the numbers $\sqrt{\lambda_j}$ are the Schmidt coefficients.*

Proof. Let $\rho_1 := \mathrm{tr}_K(|\Psi\rangle\langle\Psi|)$, where $|\Psi\rangle\langle\Psi|$ is the rank one operator, defined more generally by[4] $|x\rangle\langle y|(z) = (y|z)x$. ρ_1 is a positive operator so it can be written as $\rho_1 = \sum_n \lambda_n |\phi_n\rangle\langle\phi_n|$ for some orthonormal basis $\{\phi_n\}$ of H and scalars $\lambda_n \geq 0$. Let $\{\psi'_k\}$ be an orthonormal basis for K and write

$$\Psi = \sum_{j,k} (\phi_j \otimes \psi'_k|\Psi)\, \phi_j \otimes \psi'_k = \sum_j \phi_j \otimes \psi''_j$$

where $\psi''_j = \sum_k (\phi_j \otimes \psi'_k|\Psi)\psi'_k$. Then for arbitrary $A \in B(H)$,

$$\begin{aligned}
\sum_n \lambda_n(\phi_n|A\phi_n) &= \mathrm{tr}([\sum_n \lambda_n|\phi_n\rangle\langle\phi_n|]A) = \mathrm{tr}(\rho_1 \cdot A) \\
&= \mathrm{tr}([|\Psi\rangle\langle\Psi|] \cdot A \otimes 1) = \mathrm{tr}(|\Psi\rangle\langle(A \otimes 1)\Psi|) \\
&= (\Psi|(A \otimes 1)\Psi) = (\textstyle\sum_j \phi_j \otimes \psi''_j| \sum_k A\phi_k \otimes \psi''_k) \\
&= \sum_{j,k} (\phi_j|A\phi_k)(\psi''_j|\psi''_k).
\end{aligned}$$

[2]This space is denoted (accurately, by finite dimensionality) by $B(H \otimes K)$ in Keyl's notes
[3]The tr on the left side is the trace on H and the tr on the right side is the trace on $H \otimes K$
[4]In physics notation, $|x\rangle\langle y|(z) = |x\rangle(y|z)$

Since A is arbitrary, we have $(\psi''_j|\psi''_k) = \delta_{j,k}\lambda_j$ and therefore

$$\Psi = \sum_j \lambda_j^{1/2}\phi_j \otimes (\psi''_j/\lambda_j^{1/2}),$$

proving the existence.[5] □

Corollary 1.2 (Corollary 2.3, page 445) *Each state $\rho \in B(H)^*$ can be extended to a pure state Ψ on a larger system with Hilbert space $H \otimes H'$ such that $tr_{H'}|\Psi\rangle\langle\Psi| = \rho$. ($\Psi$ is called the "purification" of ρ.)*

Proof.[6] Let $\rho = \sum \lambda_j |\phi_j\rangle\langle\phi_j|$ be the spectral decomposition of ρ, where $\{\phi_j\}$ is an orthonormal set in H, and let ψ_j be an orthonormal basis for H', a Hilbert space of dimension at least the dimension of H. We wish to have a $\Psi \in H \otimes H'$ with $\|\Psi\| = 1$ and $\mathrm{tr}_{H'}(|\psi\rangle\langle\psi|) = \rho$.

Define $\Psi = \sum \sqrt{\lambda_j}\phi_j \otimes \psi_j$. Then

$$(\Psi|\Psi) = \sum_{j,k} (\phi_j \otimes \psi_j|\phi_k \otimes \psi_k)\sqrt{\lambda_j\lambda_k} = \sum_j \|\phi_j\|^2\|\psi_j\|^2\lambda_j = \mathrm{tr}\,(\rho) = 1.$$

For $A \in B(H)$, we need to prove that $\mathrm{tr}(\rho A) = \mathrm{tr}(|\Psi\rangle\langle\Psi|(A \otimes 1))$. The left side is equal to $\mathrm{tr}(\sum \lambda_j|\phi_j\rangle\langle A\phi_j|) = \sum \lambda_j(\phi_j|A\phi_j)$ and the right side is equal to $\mathrm{tr}(|\Psi\rangle\langle(A \otimes 1)\Psi|) = (\Psi|(A \otimes 1)\Psi) = (\sum \sqrt{\lambda_j}\phi_j \otimes \psi_j|\sum \sqrt{\lambda_k}A\phi_k \otimes \psi_k) = \sum_{j,k} \sqrt{\lambda_j\lambda_k}(\phi_j|A\phi_k)(\psi_j|\psi_k) = \sum \lambda_j(\phi_j|A\phi_j)$. □

CONVENTION: We are now going to use capital letters like $A, B\ldots$ to denote algebras of operators and lower case letters like $a, b\ldots$ will denotes the operators belonging to these algebras.

Let ρ be a state of the composite system $A \otimes B$. Here A and B denote either $B(H)$, a quantum system, or $C(X)$, a classical system. The *restriction* of ρ to A is given by $\rho^A(a) = \rho(a \otimes 1)$. The *restriction* of ρ to B is given by[7] $\rho^B(b) = \rho(1 \otimes b)$.

Trivial examples:

(1) if $\rho = \rho_1 \otimes \rho_2$ is a product state, then $\rho^A = \rho_1$ and $\rho^B = \rho_2$ (since $\rho^A(a) = \rho(a \otimes 1) = \rho_1(a)\rho_2(1)$).

(2)If both systems are quantum, then ρ^A and ρ^B are partial traces. The trick here is to identify an operator with its role as a functional. The functional ρ^A acting on $a \in A = B(H)$ is given by $\rho^A(a) = \mathrm{tr}(\rho^A a)$. The functional ρ acting on $a \otimes 1 \in A \otimes B$ with $B \in B(K)$ is given by $\rho(a \otimes 1) = \mathrm{tr}(\rho \cdot a \otimes 1)$. Hence ρ^A is a partial trace of ρ.

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[5] I am temporarily ignoring the uniqueness

[6] Not explicit in Keyl's notes, nor mentioned in the lecture

[7] A harmless notational inconsistency

Definition 1.3 (Definition 2.4, page 446) *A state ρ of a bipartite system $A \otimes B$ is called correlated if there are some $a \in A, b \in B$ such that $\rho(a \otimes b) \neq \rho^A(a)\rho^B(b)$.*

Proposition 1.4 (Proposition 2.5, page 446) *Each state ρ of a composite system $A \otimes B$ consisting of a classical system ($A = C(X)$) and an arbitrary system ($B = B(H)$) has the form*[8]

$$\rho = \sum_{j \in X} \lambda_j \rho_j^A \otimes \rho_j^B,$$

with positive weights $\lambda_j > 0$ and $\rho_j^A \in S(A)$, $\rho_j^B \in S(B)$. ($S(A)$ denotes the states of A)

Proof. Write each element of A as follows: $a = \sum_j \alpha_j |j\rangle\langle j|$. In other words, considering a as a function on $X = \{1, \ldots, n\}$, $\alpha_j = a(j)$; and thinking of a as on operator on a Hilbert space with basis $\{|j\rangle\}$, $a|j\rangle = \alpha_j |j\rangle$.

Now given $\rho \in S(A \otimes B)$, define $\rho_j^A(a) = \mathrm{tr}(a \cdot |j\rangle\langle j|) = \alpha_j$ and $\rho_j^B(b) = \lambda_j^{-1} \rho(|j\rangle\langle j| \otimes b)$, where $\lambda_j = \rho(|j\rangle\langle j| \otimes 1)$.

Obviously $\rho_j^A(1_A) = 1 = \|\rho_j^A\|$ and $\rho_j^B(1) = 1$. Furthermore, if $b \geq 0$, then $\rho(|j\rangle\langle j| \otimes b) \geq 0$ and $\rho_j^B(b) \geq 0$. Finally, for $(a, b) \in A \times B$,

$$\sum \lambda_j \rho_j^A(a) \rho_j^B(b) = \sum \lambda_j \alpha_j \lambda_j^{-1} \rho(|j\rangle\langle j| \otimes b) = \rho(\sum \alpha_j |j\rangle\langle j| \otimes b) = \rho(a \otimes b). \square$$

OFF THE WALL OBSERVATION:
If $\rho = \sum \lambda_j \rho_j^A \otimes \rho_j^B \in S(C(X) \otimes B(H))$, then the restrictions are

$$\rho^A(a) = \rho(a \otimes 1) = \sum \lambda_j \rho_j^A(a) \rho_j^B(1_B) = \sum \lambda_j \rho_j^A(a)$$

and

$$\rho^B(b) = \rho(1 \otimes b) = \sum \lambda_j \rho_j^A(1) \rho_j^B(b) = \sum \lambda_j \rho_j^B(b).$$

Definition 1.5 (Definition 2.6, page 447) *A state ρ of a composite system $B(H_1) \otimes B(H_2)$ is called separable or classically correlated if it can be written as $\rho = \sum_j \lambda_j \rho_j^{(1)} \otimes \rho_j^{(2)}$ with states $\rho_j^{(k)}$ of $B(H_k)$ and weights $\lambda_j > 0$. Otherwise ρ is called entangled.*

REMARK: Entangled states are studied through Bell inequalities. We will take this up later. We turn now to the study of completely positive maps, which are described physically as “channels”.

[8] Another notational warning: ρ_j^A is not the restriction of a state to A, it is simply a state of A

1.2 Channels (Keyl 2.3)

We learned the following theorem in Kindergarden. This (now) simple result was the starting point for the subject now called "operator spaces", or "quantized functional analysis."

Theorem 1.6 (Theorem 2.8, page 450—Stinespring dilation) *Every completely positive map $T : B(H_1) \to B(H_2)$ has the form*

$$T(a) = V^*(a \otimes 1_K)V,$$

with an additional Hilbert space K and an operator $V : H_2 \to H_1 \otimes K$. Both K and V can be chosen such that $\{(a \otimes 1)V\phi : a \in B(H_1), \phi \in H_2\}$ is total in $H_1 \otimes K$. This decomposition is unique up to unitary equivalence and is called the minimal decomposition. If $\dim H_1 = d_1$ and $\dim H_2 = d_2$, the minimal K satisfies $\dim K \leq d_1^2 d_2$.

Corollary 1.7 (Corollary 2.9, page 450—Kraus form) *Every completely positive map $T : B(H_1) \to B(H_2)$ can be written in the form*

$$T(a) = \sum_{j=1}^{N} V_j^* a V_j$$

with operators $V_j : H_2 \to H_1$ and $N \leq \dim H_1 \dim H_2$.

Proof. It is well-known that every irreducible representation of the compact operators is unitarily equivalent to the identity representation. Let $\pi(a) = a \otimes 1$ which is a representation of $B(H_1)$ on $H_2 \otimes K$. By the result just quoted, π is the direct sum $\oplus\pi_k$ of representations equivalent to the identity representation, that is, $H_1 \otimes K = \oplus_k K_k$ and $\pi(a) = \oplus_k(a|_{K_k})$. Note that we may identify K_k with H_1 by this equivalence. If P_k denotes the projection of $H_1 \otimes K$ onto K_k (or equivalently, onto H_1), then $\pi(a) = \sum_k P_k a P_k$ so that $T(a) = V^*\pi(a)V = \sum_k (V^*P_k)a(P_kV)$. □

Theorem 1.8 (Theorem 2.10, page 451) *Let ρ be a density operator on $H \otimes H_1$. Then there is a Hilbert space K, a pure state σ on $H \otimes K$ and a channel $T : B(H_1) \to B(K)$ with*

$$\rho = (Id \otimes T^*)\sigma, \tag{1}$$

where Id denotes the identity map on $B(H)^*$. The pure state σ can be chosen such that $tr_H(\sigma)$ has no zero eigenvalue. In this case, T and σ are uniquely determined up to unitary equivalence by (1), that is, if $\tilde{\sigma}, \tilde{T}$ with $\rho = (Id \otimes \tilde{T}^*)\tilde{\sigma}$ are given, we have $\tilde{\sigma} = (1 \otimes U)^*\sigma(1 \otimes U)$ and $\tilde{T}(\cdot) = U^*T(\cdot)U$ with an appropriate unitary operator U.

Proof. Let's repeat the constructions explicit in the proofs of Proposition 1.1 and Corollary 1.2.

- If $\Psi \in H \otimes K$, write $\mathrm{tr}_K(|\Psi\rangle\langle\Psi|) = \sum_n \lambda_n |\phi_n\rangle\langle\phi_n|$ with ϕ_n an ONB for H and let ψ'_k be an ONB for K and define $\psi_j = \lambda_j^{-1/2} \sum_k (\phi_j \otimes \psi'_k|\Psi)\psi'_k$. Then $\Psi = \sum_j \lambda_j^{1/2} \phi_j \otimes \psi_j$.
- If ρ is a state of $B(H)$, say $\rho = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$ with ϕ_j and ONB for H and if ψ_j is an ONB for H' then $\Psi := \sum \lambda_j^{1/2} \phi_j \otimes \psi_j$ satisfies $\rho = \mathrm{tr}_{H'}(|\Psi\rangle\langle\Psi|)$ (Ψ is the "purification" of ρ)

Turning to the construction for the proof of Theorem 1.8, let σ be the "purification" of $\mathrm{tr}_{H_1}(\rho)$, so $\exists K, \exists \Psi \in H \otimes K$ with $\mathrm{tr}_{H_1}(\rho) = \mathrm{tr}_K(|\Psi\rangle\langle\Psi|)$. By Proposition 1.1, $\Psi = \sum \lambda_j^{1/2} \psi_j \otimes \phi_j$ where λ_j, ψ_j is the "spectral data" for $\mathrm{tr}_{H_1}(\rho)$ and ϕ_j is an ONB for K. Now the operator $T : B(H_1) \to B(K)$ is determined by applying each side of (1) to $|\psi \otimes \eta_k\rangle\langle\psi_l \otimes \eta_p|$ (which equals $(|\psi_j\rangle\langle\psi_l|) \otimes (|\eta_k\rangle\langle\eta_l|)$), where η_k is an ONB for H_1, which results in

$$\rho(|\psi \otimes \eta_k\rangle\langle\psi_l \otimes \eta_p|) = (\lambda_l \lambda_j)^{1/2} (\phi_j | T(|\eta_k\rangle\langle\eta_p|)\phi_l). \qquad (2)$$

Here is the calculation of the right side of (1):

$$\begin{aligned}
(\mathrm{Id} \otimes T^*)\sigma(|\psi \otimes \eta_k\rangle\langle\psi_l \otimes \eta_p|) &= \sigma(\mathrm{Id} \otimes T((|\psi_j\rangle\langle\psi_l|) \otimes (|\eta_k\rangle\langle\eta_l|)) \\
&= (\Psi|\mathrm{Id} \otimes T((|\psi_j\rangle\langle\psi_l|) \otimes (|\eta_k\rangle\langle\eta_l|)\Psi) \\
&= (\Psi|[|\psi_j\rangle\langle\psi_l|) \otimes T(|\eta_k\rangle\langle\eta_l|)]\Psi) \\
&= \sum_{\alpha,\beta} (\lambda_\alpha \lambda_\beta)^{1/2} \delta_{\beta j} (\psi_\alpha \otimes \phi_\alpha | \psi_l \otimes T(|\eta_k\rangle\langle\eta_p|)\phi_\beta) \\
&= (\lambda_l \lambda_j)^{1/2} (\phi_j | T(|\eta_k\rangle\langle\eta_p|)\phi_l).
\end{aligned}$$

It remains to show that T is completely positive. This consists of two steps, which are going to be left as exercises (more precisely, gaps).

- (EXERCISE) Insert $\rho = |\chi\rangle\langle\chi|$ (with $\chi \in H \otimes H_1$) into (2) to get $Ta = V^* a V$ where $(V\phi_j|\eta_k) = \lambda_j^{-1/2}(\psi_j \otimes \eta_k|\chi)$
- (EXERCISE) Prove the case for general ρ by writing $\rho = \sum \mu_j |\chi_j\rangle\langle\chi_j|$, with $\mu_j \geq 0$ and $\sum \mu_j = 1$.

2 February 6, 2007—Keyl, Section 2—Basic Concepts—continued

2.1 Separability Criteria and Positive Maps (Keyl 2.4)

Lemma 2.1 (Proposition 2.12, page 452) *For any entangled state $\rho \in S(H \otimes K)$ there is an operator a on $H \otimes K$, called entanglement witness for ρ with the property $\rho(a) < 0$ and $\sigma(a) \geq 0$ for all separable $\sigma \in S(H \otimes K)$.*

Proof. The set D of separable states is a closed convex set[9], and $\rho \notin D$, so $\exists$ a linear functional α and $\gamma \in \mathbf{R}$ such that $\alpha(\rho) < \gamma \leq \alpha(\sigma)$ for all $\sigma \in D$. By replacing α by $\alpha - \gamma \mathrm{tr}(\cdot)$ we may assume that $\gamma = 0$. Since $\alpha(a) = \mathrm{tr}(a\sigma)$ for some $a \in B(H \otimes K)$, the lemma is proved. □

For the proof of the next theorem, we need the results of reference [94] in Keyl, which is summarized as follows:

Let the inner products on

$$H_1,\ H_2,\ H_1 \otimes H_2,\ A_1 := B(H_1),\ A_2 := B(H_2),\ A_1 \otimes A_2$$

be denoted respectively by

$$(\cdot,\cdot)_1,\ (\cdot,\cdot)_2,\ ((\cdot,\cdot)),\ [a,b]_1 = \mathrm{tr}(b^* a),\ [\cdot,\cdot]_2,\ [[\cdot,\cdot]].$$

Define a linear operator $J : B(A_1, A_2) \to A_1 \otimes A_2$ by the rule

$$[[J(T), a^* \otimes b]] = [T(a), b] \text{ for } a \in A_1, b \in A_2 \text{ and } T \in B(A_1, A_2).$$

John dePillis proved in 1967 (Pacific Journal of Mathematics) that

- $J(T) = \sum_i e_i^* \otimes T(e_i)$ where (e_i) is any ONB of A_1
- T maps hermitian operators to hermitian operators if and only if $J(T)$ is hermitian.

Now the main results of [94] are

- Let $T \in B(A_1, A_2)$. Then T is a positive operator if and only if

  $$((J(T)(x \otimes y), x \otimes y)) \geq 0 \text{ for all } x \in H_1, y \in H_2,$$

  where $x \otimes y$ is the operator $z \mapsto (z, y)_1 x$.

- Let $T \in B(A_1, A_2)$ and suppose that T maps hermitian operators to hermitian operators. Then T preserves trace if and only if

  $$\sum_k ((J(T)(e_i \otimes f_k), e_j \otimes f_k)) = \delta_{ij} \text{ for all ONBs } (e_k) \text{ in } H_1 \text{ and } (f_i) \text{ in } H_2.$$

Theorem 2.2 (Theorem 2.11, page 452) *A state $\rho \in B(H \otimes K)^*$ is separable if and only if for any positive map $T^* : B(K)^* \to B(H)^*$ the operator $(Id \otimes T^*)\rho$ is positive.*

[9] EXERCISE: Show it is a closed set

Proof. If ρ is separable, it is the sum of product states $\rho_1 \otimes \rho_2$, and since for any positive map $T^* : B(K)^* \to B(H)^*$, $(\mathrm{Id} \otimes T^*)\rho_1 \otimes \rho_2$ is positive, it follows that $(\mathrm{Id} \otimes T^*)\rho$ is positive.

Conversely, for $a \in B(H \otimes K)$, define $T_a^* : B(K)^* \to B(H)^*$ by

$$\mathrm{tr}(a \cdot \rho_1 \otimes \rho_2) = \mathrm{tr}(\rho_1^t \cdot T_a^*(\rho_2)).$$

For the record, the trace on the left side is the trace on $H \otimes K$, $\rho_1 \otimes \rho_2$ is a product state in $B(H \otimes K)^*$, the trace on the right side is the trace on H, and the transpose is with respect to an arbitrary but fixed ONB $\{|j\rangle : j = 1, \ldots, d\}$ of H.

CLAIM 1: If a is such that $\mathrm{tr}(a \cdot \rho_1 \otimes \rho_2) \geq 0$ for all product states $\rho_1 \otimes \rho_2$, then T_a^* is a positive operator.

EXERCISE: Prove CLAIM 1 using [94].

CLAIM 2: For $a \in B(H \otimes K)$, $\rho \in B(H \otimes K)^*$ and $\Psi = d^{-1/2} \sum_j |j\rangle \otimes |j\rangle \in H \otimes H$, then

$$\mathrm{tr}(a \cdot \rho) = \mathrm{tr}((|\Psi\rangle\langle\Psi|)[(\mathrm{Id} \otimes T_a^*)(\rho)]).$$

EXERCISE: Prove CLAIM 2 (it is just a calculation!).

We can now complete the proof of Theorem 2.2. Assume that $(\mathrm{Id} \otimes T^*)\rho$ is positive for all positive $T^* : B(K)^* \to B(H)^*$. By claims 1 and 2, $\mathrm{tr}(a \cdot \rho) \geq 0$ provided $\mathrm{tr}(a \cdot \rho_1 \otimes \rho_2) \geq 0$ for all product states and hence all separable states. By Lemma 2.1 we are done, for if ρ is entangled, then $\exists a$ with $\mathrm{tr}(\rho \cdot a) < 0$ and $\mathrm{tr}(a \cdot \rho_1 \otimes \rho_2) \geq 0$. □

For the proof of the next theorem, we need the results of reference [174] in Keyl, which is summarized as follows:

A linear map $\Phi : A \to B$ of C^*-algebras is n-positive if $\Phi \otimes \mathrm{Id} : A \otimes M_n \to B \otimes M_n$ is positive, and completely positive (CP) if it is n-positive for all $n \geq 1$. It is a theorem of Stormer (Lecture Notes in Physics **29**, 1974) that $CP(M_n, M_m)$ is the convex cone generated by the maps $M_n \ni a \mapsto s^* a s \in M_m,\ s \in M_{n \times m}$.

A linear map $\Phi : A \to B$ of C^*-algebras is n-copositive if $\Phi \otimes T : A \otimes M_n \to B \otimes M_n$ is positive, and completely copositive (CCP) if it is n-copositive for all $n \geq 1$. It is also true that $CCP(M_n, M_m)$ is the convex cone generated by the maps $M_n \ni a \mapsto s^* a^T s \in M_m,\ s \in M_{n \times m}$.

A natural question to ask in this context is: Is every positive map from A to B is a sum of CP and CCP maps. By results of Stormer and Woronowicz of the early 60s, the answer is yes for $(A, B) = (M_2, M_2)$, and by a result of Choi, the answer is no for $(A, B) = (M_3, M_3)$. The main results of [174] are that the answer is yes for $(A, B) = (M_2, M_3)$ and no for $(A, B) = (M_2, M_4)$, from which it follows that the answer is yes for $(A, B) = (M_3, M_2)$ and no for $(A, B) = (M_4, M_2)$.

Theorem 2.3 (Theorem 2.13, page 453) *Consider a bipartite system* $B(H \otimes K)$ *with* $\dim H = 2$ *and* $\dim K = 2$ *or* 3*. A state* $\rho \in S(H \otimes K)$ *is separable if and only if its partial transpose is positive.*

EXERCISE: Supply the proof of Theorem 2.3 from reference [86] in Keyl using reference [174].[10]

Proposition 2.4 (Proposition 2.15, page 453) *Each ppt-state* $\rho \in S(H \otimes K)$ *satisfies the reduction criterion. If* $\dim H = 2$ *and* $\dim K = 2$ *or* 3*, both criteria are equivalent.*

EXERCISE: Supply the proof (and the definitions!) of Proposition 2.4 from references [85] and [42] in Keyl.

[10] Reference [86] also contains a proof of Theorem 2.2, which may be more illuminating than the one (partially) provided here