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On classification problem of Loday algebras

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Abstract. This is a survey paper on classification problems of some classes of algebras introduced by Loday around 1990s. In the paper the author intends to review the latest results on classification problem of Loday algebras, achievements have been made up to date, approaches and methods implemented.

1. Introduction

It is well known that any associative algebra gives rise to a Lie algebra, with bracket $[x, y] := xy - yx$. In 1990s J.-L. Loday introduced a non-antisymmetric version of Lie algebras, whose bracket satisfies the Leibniz identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

and therefore they have been called Leibniz algebras. The Leibniz identity combined with antisymmetry, is a variation of the Jacobi identity, hence Lie algebras are antisymmetric Leibniz algebras. The Leibniz algebras are characterized by the property that the multiplication (called a bracket) from the right is a derivation but the bracket no longer is skew-symmetric as for Lie algebras. Further Loday looked for a counterpart of the associative algebras for the Leibniz algebras. The idea is to start with two distinct operations for the products xy, yx, and to consider a vector space D (called an associative dialgebra) endowed by two binary multiplications $\dashv$ and $\vdash$ satisfying certain "associativity conditions". The conditions provide the relation mentioned above replacing the Lie algebra and the associative algebra by the Leibniz algebra and the associative dialgebra, respectively. Thus, if $(D, \vdash, \dashv)$ is an associative dialgebra, then $(D, [x, y] = x \dashv y - y \vdash x)$ is a Leibniz algebra. The functor $(D, \vdash, \dashv) \longrightarrow (D, [x, y])$ has a left adjoint, the algebra $(D, \vdash, \dashv)$ is the universal enveloping dialgebra of the Leibniz algebra $(D, [x, y])$. The Kozsul dual of the associative dialgebras are algebras (called dendriform algebras) possessing two operations $\prec$ and $\succ$ such that the product made of the sum $x \prec y + y \succ x$ is associative. Loday has given the explicit description of the free dendriform algebras by means of binary trees and constructed the (co)homology groups for dendriform algebras which, as in the case of dialgebras, vanish on the free objects. The class of dendriform algebras with dual-Leibniz algebras and associative algebras based on the relationship between binary trees and permutations.

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The (co)homology theory for dialgebras has been constructed by Loday. As it was mentioned above he also has proved that the (co)homology groups vanish on the free dialgebra. As a consequence, one gets a new approach to the (co)homology theory for ordinary associative algebras. The surprising fact is that in the construction of the chain complex of the new classes of algebras the combinatorics of planar binary trees is involved.

The purpose of this article is to review the classification results on four classes of algebras introduced by Loday: associative dialgebras, dendriform algebras, Leibniz and Zinbiel algebras. In fact, much attention is paid on algebraic classification problem of Leibniz algebras along with a brief review on classification of other classes of Loday algebras, whereas the most (co)homological results, applications of Loday algebras, the geometric classification problem, that is finding generic structural constants in the sense of algebraic groups, the rigidity problems and finding Lie-group like objects, and others are beyond the scope of the paper. The problems related to the group theoretical realizations and integrability problems of Leibniz algebras are studied by Kinyon and Weinstein [**42**]. Deformation theory of Leibniz algebras and related physical applications of it are initiated by Fialowski, Mandal, Mukherjee [**32**]. The notion of simple Leibniz algebra was suggested by Dzhumadil'daev in [**29**]

2. Loday diagram

2.1. Leibniz algebras: appearance. It is well-known that the Chevalley-Eilenberg chain complex of a Lie algebra $\mathfrak{g}$ is the sequence of chain modules given by the exterior powers of $\mathfrak{g}$

$$\bigwedge * \,\mathfrak{g}: \quad ... \longrightarrow \wedge^{n+1}\mathfrak{g} \xrightarrow{d_{n+1}} \wedge^{n}\mathfrak{g} \xrightarrow{d_n} \wedge^{n-1}\mathfrak{g} \xrightarrow{d_{n-1}} ...$$

and the boundary operators $d_n : \wedge^n\mathfrak{g} \longrightarrow \wedge^{n-1}\mathfrak{g}$ classically defined by

$$d_n(x_1 \wedge x_2 \wedge ... \wedge x_n) := \sum_{i<j} (-1)^{n-j} x_1 \wedge ... \wedge x_{i-1} \wedge \widehat{[x_i, x_j]} \wedge ... \wedge x_{j-1} \wedge \hat{x_j} \wedge ... \wedge x_n.$$

The property $d \circ d = 0$, which makes this sequence a chain complex, is proved by using the antisymmetry $x \wedge y = -y \wedge x$ of the exterior product, the Jacobi identity $[[x, y], z] + [[y, z], x] + [z, x], y] = 0$ and the antisymmetry $[x, y] = -[y, x]$ of the bracket on $\mathfrak{g}$.

Let us consider the following chain complex replacing in the above chain complex of the Lie algebra $\mathfrak{g}$ by an algebra L and the exterior product $\wedge$ by the tensor product $\otimes$, respectively:

$$... \longrightarrow L^{\otimes(n+1)} \xrightarrow{d_{n+1}} L^{\otimes n} \xrightarrow{d_n} L^{\otimes(n-1)} \xrightarrow{d_{n-1}} ...$$

If one rewrites the boundary operator as

$$d_n(x_1 \otimes x_2 \otimes ... \otimes x_n) = \sum_{1 \leq i<j \leq n} (-1)^{n-j} x_1 \otimes ... \otimes x_{i-1} \otimes \widehat{[x_i, x_j]} \otimes ... \otimes x_{j-1} \otimes \hat{x_j} \otimes ... \otimes x_n,$$

then the property $d \circ d = 0$ is proved without making use of the antisymmetricity properties of both the exterior product and the binary operation (bracket) on the algebra L, it suffices that the bracket on L satisfies the so called Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

This was a motivation to introduce a new class of algebras called Leibniz algebras by J.-L. Loday in 1990s. In fact, Leibniz algebras have been introduced in the mid-1960's by Bloh under the name D-algebras [**16**]. They appeared again after Loday's work [**44**], where they have been called Leibniz algebras. Thus a Leibniz algebra L is a vector space over a field K equipped with a bilinear map

$$[\cdot,\cdot] : L \times L \longrightarrow L$$

satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad \forall x,y,z \in L.$$

If $d_z(\cdot) = [\cdot, z]$ then the Leibniz identity is written as $d_z([x,y]) = [d_z(x), y] + [x, d_z(y)]$ which is the Leibniz rule for the operator d_z, where $z \in L$.

During the last 20 years the theory of Leibniz algebras has been actively studied and many results on Lie algebras have been extended to Leibniz algebras.

The (co)homology theory, representations and related problems of Leibniz algebras have been studied by Loday himself, his students and colleagues [**47**], [**46**]. Most developed part of the class of Leibniz algebras is its solvable and nilpotent parts (and their subclasses, later on we review the results on classification of these subclasses). However, in 2011 the analogue of Levi's theorem was proved by D. Barnes [**14**]. He showed that any finite-dimensional complex Leibniz algebra is decomposed into a semidirect sum of its solvable radical and a semisimple Lie algebra. As it is well-known that the semisimple part can be decomposed into a direct sum of simple Lie algebras and therefore the main issue in the classification problem of finite-dimensional complex Leibniz algebras is reduced to the study of the solvable part. Therefore, the classification of solvable Leibniz algebras is important to construct finite-dimensional Leibniz algebras.

2.2. Associative dialgebras: enveloping algebras of Leibniz algebras. We start with a generalization of associative algebras called *associative dialgebras* by Loday. **Diassociative algebra** D is a vector space equipped with the two bilinear binary operations:

$$\dashv: D \times D \to D \text{ and } \vdash: D \times D \to D$$

satisfying the axioms

$$\begin{aligned}(x \vdash y) \vdash z &= (x \dashv y) \vdash z,\\ (x \vdash y) \dashv z &= x \vdash (y \dashv z),\\ x \dashv (y \vdash z) &= x \dashv (y \dashv z).\end{aligned}$$

for all $x, y, z \in D$.

The classification of the associative dialgebras in low-dimensions has been given by using the structure constants and a Computer program in Maple [**15**], [**62**], [**63**].

2.3. Category of Zinbiel algebras: Kozsul dual of Leibniz algebras. Loday defined a class of Zinbiel algebras, which is Koszul dual to the category of Leibniz algebras as follows.

Definition 2.1. *Zinbiel algebra R is an algebra with a binary operation $\cdot$: $R \times R \to R$, satisfying the condition:*

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) + a \cdot (c \cdot b)$$

A.S. Dzhumadildaev et al. proved that any finite-dimensional Zinbiel algebra is nilpotent and have given the lists of isomorphism classes of Zinbiel algebras in dimension 3 [**30**]. Early 2-dimensional case has been classified by B. Omirov [**51**]. Further classifications have been carried out by using the isomorphism invariant called characteristic sequence [**3**], [**4**], [**5**], [**17**] (the characteristic sequences have been successfully used in Lie algebras case).

2.4. Category of Dendriform algebras: Kozsul dual of associative dialgebras.

Definition 2.2. *Dendriform algebra E is an algebra with two binary operations*

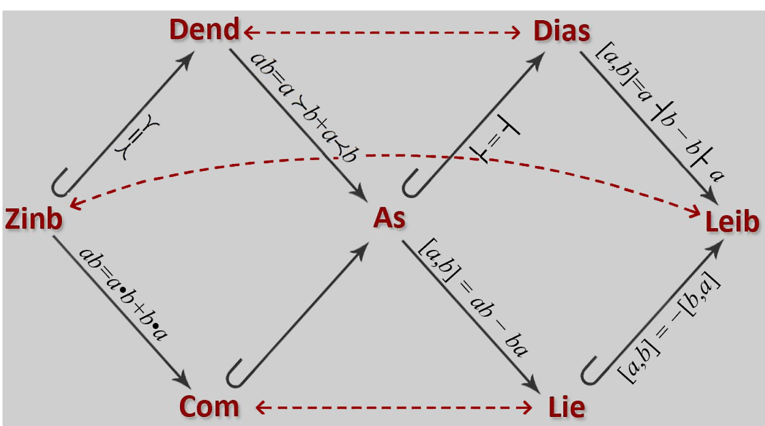
$$\succ: E \times E \to E, \quad \prec: E \times E \to E$$

satisfying the following axioms:

$$\begin{gathered}(a \prec b) \prec c = (a \prec c) \prec b + a \prec (b \succ c),\\ (a \succ b) \prec c = a \succ (b \prec c),\\ (a \prec b) \succ c + (a \succ b) \succ c = a \succ (b \succ c).\end{gathered}$$

The essential results on Dendriform algebras have been obtained by Aguiar, Guo and Ebrahimi-Fard [**6**], [**31**], [**37**]. There is classification of two-dimensional dendriform algebras [**70**].

2.5. Loday Diagram and Kozsul duality. The results intertwining Loday algebras are best expressed in the framework of algebraic operads. The notion of diassociative algebra defines an algebraic operad **Dias**, which is binary and quadratic. According to the theory of Ginzburg and Kapranov there is a well-defined "dual operad" **Dias**$^!$. Loday has showed that this is exactly the operad **Dend** of the dendriform algebras, in other words a dual diassociative algebra is nothing but a dendriform algebra. The similar duality can be established between the algebraic operads **Leib** defined by the notion of Leibniz algebra and the algebraic operads **Zinb** defined by the notion of Zinbiel algebra. Operadic dualities **Com**$^!$ = **Lie**, **As**$^!$ = **As** have been proved in [**33**], **Dias**$^!$ = **Dend** is in [**46**] and **Leib**$^!$ = **Zinb** is in [**45**]. The categories of algebras over these operads assemble into a commutative diagram of functors which reflects the Koszul duality (see [**46**]).



Picture 1. Loday Diagram and Kozsul Duality.

3. Leibniz algebras: Structural Theory

3.1. More on Leibniz algebras. In fact, the definition of the Leibniz algebras given above is a definition of the right Leibniz algebras, whereas the identity for the left Leibniz algebra is as follows

$$[x,[z,y]] = [[x,z],y] + [z,[x,y]], \text{ for all } x,y,z \in L.$$

The passage from the right to the left Leibniz algebra can be easily done by considering a new product "$[\cdot,\cdot]_{opp}$" on the algebra by "$[x,y]_{opp} = [y,x]$." Clearly, a Lie algebra is a Leibniz algebra, and conversely, a Leibniz algebra L with property $[x,y] = -[y,x]$, for all $x,y \in L$ is a Lie algebra. Hence, we have an inclusion functor $inc : Lie \longrightarrow Leib$. This functor has a left adjoint $imr : Leib \longrightarrow Lie$ which is defined on the objects by $L_{Lie} = L/I$, where I is the ideal of L generated by all squares. That is any Leibniz algebra L gives rise to the Lie algebra L_{Lie}, which is obtained as the quotient of L by the relation $[x,x] = 0$. The I is the minimal ideal with respect to the property that $\mathfrak{g} := L/I$ is a Lie algebra. The quotient mapping $\pi : L \longrightarrow \mathfrak{g}$ is a homomorphism of Leibniz algebras. One has an exact sequence of Leibniz algebras:

$$0 \longrightarrow I \longrightarrow L \longrightarrow L_{Lie} \longrightarrow 0.$$

We consider finite-dimensional algebras L over a field K of characteristic 0 (in fact it is only important that this characteristic is not equal to 2). A linear transformation d of a Leibniz algebra L is said to be a derivation if $d([x,y]) = [d(x),y] + [x,d(y)]$ for all $x,y \in L$. The set of all derivations of L is denoted by $Der(L)$. Due to the operation of commutation of linear operators $Der(L)$ is a Lie algebra. Let us consider $d_a : L \longrightarrow L$ defined by $d_a(x) = [x,a]$ for $a \in L$. Then the Leibniz identity is written as $d_a([x,y]) = [d_a(x),y] + [x,d_a(y)]$ for any $a,x \in L$ showing that the operator d_a for all $a \in L$ is a derivation on the Leibniz algebra L. In other words, the right Leibniz algebra is characterized by this property, i.e., any right multiplication operator is a derivation of L. Notice that for the left Leibniz algebras a left multiplication operator is a derivation. The theory of Leibniz algebras was developed by Loday himself with his coauthors. Mostly they dealt with the (co)homological problems of this class of algebras. The study of structural properties of Loday algebras has begun after private conversation between Loday and Ayupov in Strasbourg in 1994.

The automorphism group $Aut(L)$ of a Leibniz algebra L can be naturally defined. If the field K is $\mathbb{R}$ or $\mathbb{C}$, then the automorphism group is a Lie group and the Lie algebra $Der(L)$ is its Lie algebra. One can consider $Aut(L)$ as an algebraic group (or as a group of K-points of an algebraic group defined over the field K). Note that those results proven for the right Leibniz algebras $(L,\cdot)$ can be easily rewritten for the left Leibniz algebras $(L,\star)$ by using the $x \star y = y \cdot x$, and another way around. Also note that for the right Leibniz algebras the set of all right multiplications r_x form a Lie subalgebra ($[r_y,r_x] = r_y \circ r_x - r_x \circ r_y = r_{[x,y]}$) of $DerL$ denoted by $ad(L)$.

Let L be any right Leibniz algebra. Consider a subspace spanned by elements of the form $[x,x]$ for all possible $x \in L$ denoted by $I : I = Span_K\{[x,x] |\ x \in L\}$. (Loday has denoted it by L^{ann} and another term "liezator" has been used by Gorbatsevich [**35**]). In fact, I is a two-sided ideal in L. The product $[L,I]$ is equal to 0 due to the Leibniz identity. The fact that it is a right ideal follows from the identity

$$[[x,x],y] = [[y,y]+x,[y,y]+x] - [x,x] \text{ in } L.$$

For a non-Lie Leibniz algebra L the ideal I always is different from the L. The quotient algebra L/I is a Lie algebra. Therefore I can be viewed as a "non-Lie core" of the L. The ideal I can also be described as the linear span of all elements of the form $[x,y]+[y,x]$. The quotient algebra L/I is called the liezation of L and it could be denoted by L_{Lie}. There is a natural action of the Lie algebra L_{Lie} on a vector space I (multiplication in which is trivial). If the Leibniz algebra L is commutative (i.e. $[x,y]=[y,x]$,) then the subset $[x,y]+[y,x]$ coincides with the set $[L,L]=\{[x,y]\ x,y\in L\}$. But then liezation of the algebra L is just a commutative Lie algebra $L/[L,L]$. Commutative Leibniz algebras are nilpotent, their class of nilpotency is equal to 2. This kind of Leibniz algebras can be described in some detail (Gorbatsevich's suggestion!).

Let us now consider the centers of Leibniz algebras. Since there is no commutativity there are two left and right centers, which are given by

$$Z^l(L)=\{x\in L|[x,L]=0\} \text{ and } Z^r(L)=\{x\in L|[L,x]=0\},$$

respectively. Both these centers can be considered for the left and right Leibniz algebras. For the right Leibniz algebra L the right center $Z^r(L)$ is an ideal, moreover it is two-sided ideal (since $[L,[x,y]]=-[L,[y,x]]$) but the $Z^l(L)$ need not be a subalgebra. For the left Leibniz algebra it is exactly opposite. In general, the left and right centers are different; even they may have different dimensions. Obviously, $I\subset Z^r(L)$. Therefore $L/Z^r(L)$ is a Lie algebra, which is isomorphic to the Lie algebra $ad(L)$ mentioned above.

Many notions in the theory of Lie algebras may be naturally extended to Leibniz algebras. For example, the solvability is defined by the derived series:

$$D^n(L): D^1(L)=[L,L],\ D^{k+1}(L)=[D^k(L),D^k(L)],\ k=1,2,...$$

A Leibniz algebra is said to be *solvable* if its derived series terminates. It is easy to verify that the sum of solvable ideals in a Leibniz algebra also is a solvable ideal. Therefore, there exists a largest solvable ideal R containing all other solvable ideals. Naturally, it is called *the radical* of Leibniz algebra. Since the ideal I of a Leibniz algebra L is abelian it is contained in the radical R of L.

The notion of nilpotency also can be defined by using the decreasing central series

$$C^n(L): C^1(L)=[L,L],\ C^{k+1}(L)=[L,C^k(L)],\ k=1,2,3,... \text{ of } L.$$

Despite of a certain lack of symmetry of the definition (multiplication by L only from the right) members of this series are two-sided ideals, moreover, a simple observation shows that the inclusion $[C^p(L),C^q(L)]\subset C^{p+q}(L)$ is implied. Leibniz algebra is called *nilpotent* if its central series terminates. As it is followed from the definition that the centers (left and right) for nilpotent Leibniz algebras are nontrivial. Any Leibniz algebra L has a maximal nilpotent ideal containing all other nilpotent ideals of L. This ideal of L is said to be *the nilradical* of L denoted by N. The nilradical is a characteristic ideal, i.e., it remains invariant under all automorphisms of the Leibniz algebra L. Obviously, it is contained in the radical of L and it equals to the nilradical of the solvable radical of L. The nilradical contains left center, as well as the ideal I.

Linear representation (sometimes referred as module) of a Leibniz algebra is a vector space V, equipped with two actions (left and right) of the Leibniz algebra L

$$[\cdot,\cdot]: L\times V\longrightarrow V \text{ and } [\cdot,\cdot]: V\times L\longrightarrow V,$$

such that the identity

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

is true whenever one (any) of the variables is in V, and the other two in L, i.e.,

- $[[x,y],v] = [[x,v],y] - [x,[y,v]]$;
- $[[x,v],y]] = [[x,y],v] - [x,[v,y]]$;
- $[[v,x],y]] = [[v,y],x] - [v,[x,y]]$.

Note that the concept of representations of Lie algebras and Leibniz algebras are different. Therefore, such an important theorem in the theory of Lie algebras, as the Ado theorem on the existence of faithful representation in the case of Leibniz algebras was proved much easier and gives a stronger result. It is because the kernel of the Leibniz algebra representation is the intersection of kernels (in general,different one's) of right and left actions, in contrast to representations of Lie algebras, where these kernels are the same. Therefore, an faithful representation of Leibniz algebras can be obtained easier than faithful representation of the case of Lie algebras (see [**13**]).

PROPOSITION 3.1. *Any Leibniz algebra has a faithful representation of dimension no more than* $dim(L) + 1$.

Here is Barnes's result on an analogue of Levi-Malcev Theorem for Leibniz algebras.

PROPOSITION 3.2. (*Levi theorem for Leibniz algebras*) *For a Leibniz algebra* L *there exists a subalgebra* S *which gives the decomposition* $L = S \dot{+} R$, *where* R *is the radical of* L.

Barnes has given the non-uniqueness of the subalgebra S (the minimum dimension of Leibniz algebra in which this phenomena appears is 6). It is known that in the case of Lie algebras the semi-simple Levi factor is unique up to conjugation. Moreover, from the proof of the theorem it can be easily seen that S is a semisimple Lie algebra.

3.2. Universal enveloping algebra (Poincare-Birkhoff-Witt Theorem). The universal enveloping algebra for a Leibniz algebra has been constructed by Loday and Pirashvili. Loday showed that such an algebra comes in as an algebra with two bilinear binary associative operations satisfying three axioms (see Section 2.2) (the algebra has been called *associative dialgebra* by Loday). Loday and Pirashvili constructed an enveloping algebra of a Leibniz algebra L by using the concept the free diassociative algebra. In algebras over fields this construction is interpreted as follows. Let V be a vector space over a field K. By definition the free dialgebra structure on V is the dialgebra $D(V)$ equipped with a K-linear map $i : V \longrightarrow D$ such that for any K-linear map $f : V \longrightarrow D'$, where D' is a dialgebra over K, there exists a unique factorization

$$f : V \xrightarrow{i} D(V) \xrightarrow{h} D',$$

where h is a dialgebra morphism. The authors proved the existence of $D(V)$ giving it as tensor module

$$T(V) := K \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

with the two products inductively defined by

$$(v_{-n}\cdots v_{-1}\otimes v_0\otimes v_1\cdots v_m)\dashv(w_{-p}\cdots w_{-1}\otimes w_0\otimes w_1\cdots w_q)$$
$$=v_{-n}\cdots v_{-1}\otimes v_0\otimes v_1\cdots v_m w_{-p}\cdots w_q,$$

$$(v_{-n}\cdots v_{-1}\otimes v_0\otimes v_1\cdots v_m)\vdash(w_{-p}\cdots w_{-1}\otimes w_0\otimes w_1\cdots w_q)$$
$$=v_{-n}\cdot\cdots\cdot v_m w_{-p}\cdots w_{-1}\otimes w_0\otimes w_1\cdots w_q,$$

where $v_i, w_j \in V$.

The universal enveloping dialgebra of a Leibniz algebra L is defined as the following quotient of the free dialgebra on L :

$$Ud(L) := T(L)\otimes L\otimes T(L)/\{[x,y]-x\dashv y+y\vdash x|\ x,y\in L\}.$$

3.3. Solvable Leibniz algebras. Owing to a result of Mubarakzjanov [**49**] a new approach for studying the solvable Lie algebras by using their nilradicals was developed (see [**9**], [**21**], [**50**], [**73**] and [**74**]). The analogue of Mubarakzjanov's results has been applied for Leibniz algebras in [**23**] which shows the importance of the consideration of their nilradicals in Leibniz algebras case as well (also see [**22**], [**24**], [**40**] and [**41**]).

PROPOSITION 3.3. *(Lie Theorem for solvable Leibniz algebras) A solvable right Leibniz algebra L over $\mathbb{C}$ has a complete flag of subspaces which is invariant under the right multiplication.*

In other words, all linear operators r_x of right multiplications can be simultaneously reduced to triangular form.

PROPOSITION 3.4. *Let R be the radical of a Leibniz algebra L, and N be its nilradical. Then $[R, L] \subset N$.*

Propositions 3.3 and 3.4 for left Leibniz algebras have been given in [**35**].

COROLLARY 3.5. *One has $[R, R] \subset N$. In particular, $[R, R]$ is nilpotent.*

COROLLARY 3.6. *Leibniz algebra L is solvable if and only if $[L, L]$ is nilpotent.*

PROPOSITION 3.7. *Multiplications (right and left) in the Leibniz algebra are degenerate linear operators.*

3.4. Nilpotent Leibniz algebras (Engel's Theorem).

PROPOSITION 3.8. (*Engel's theorem for Leibniz algebras*) *If all operators r_x of right multiplication for the right Leibniz algebra L are nilpotent, then the algebra L is nilpotent. In particular, for right multiplications there is a common eigenvector with zero eigenvalue.*

There exists a basis with respect to that the matrices of all r_x have upper-triangular form. The notion of normalizer is defined as follows. The left and right normalizers $N^l_L(U)$ and $N^r_L(U)$ of a subset U of a Leibniz algebra L are given as follows

$$N^l_L(U)=\{x\in L|\ [x,U]\subset U\}\quad N^r_L(U)=\{x\in L|\ [U,x]\subset U\}.$$

COROLLARY 3.9. *The normalizers of a subalgebra M in a nilpotent Leibniz algebra L strictly contain M.*

COROLLARY 3.10. *A subspace* $V \subset L$ *generates a Leibniz algebra if and only if* $V + [L, L] = L$.

It is interesting to note that not all properties of nilpotent Lie algebras, even a simple and well-known one's, hold for the case of Leibniz algebras. For example, there is a simple statement for nilpotent Lie algebras of dimension 2 or more: "the codimension of the commutant is more or equal to 2". For Leibniz algebras it is not true (though not only for nilpotent, but for all solvable Leibniz algebras we have $codim_L[L, L] > 0$). For example, two-dimensional Leibniz algebra $L = span\{e_1, e_2\}$, with $[e_1, e_1] = e_2$ is nilpotent, but its commutant has codimension 1. This is due to the fact that its liezation is one-dimensional. But for one-dimensional Lie algebras above mentioned statement is incorrect. We obtain the following useful corollary.

COROLLARY 3.11. *If Leibniz algebra* L *is nilpotent and* $codim_L([L, L]) = 1$, *then the algebra* L *is generated by one element.*

So for $codim_L([L, L]) = 1$ a nilpotent Leibniz algebra is a kind of "cyclic". The study of such nilpotent algebras is the specifics of the theory of Leibniz algebras; Lie algebra has no analogue. Such cyclic L can be explicitly described (Gorbatsevich's suggestion).

COROLLARY 3.12. *The minimal number of generators of a Leibniz algebra* L *equals* $dim\ L/[L, L]$.

3.5. Filiform Leibniz algebras. Let L be a Leibniz algebra. Define

$$L^1 = L,\ L^{k+1} = [L^k, L],\ k \geq 1.$$

Clearly,

$$L^1 \supseteq L^2 \supseteq \cdots .$$

DEFINITION 3.13. *A Leibniz algebra* L *is said to be a nilpotent, if there exists* $s \in \mathbb{N}$, *such that*

$$L^1 \supset L^2 \supset ... \supset L^s = \{0\}.$$

DEFINITION 3.14. *A Leibniz algebra* L *is said to be a filiform, if* $dimL^i = n - i$, *where* $n = dimL$ *and* $2 \leq i \leq n$.

The class of filiform Leibniz algebras in dimension n over K we denote by $Lb_n(K)$. There is a breaking of $Lb_n(\mathbb{C})$ into three subclasses:

$$Lb_n(\mathbb{C}) = FLb_n \cup SLb_n \cup TLb_n.$$

Classification results. Let us now to give an up-to-date results on classification of complex Leibniz algebras. Two-dimensional Leibniz algebras have been given by Loday [**44**]. In dimension three there are fourteen isomorphism classes (5 parametric family of orbits and 9 single orbits), the list can be found in [**26**] and [**69**]. There is no simple Leibniz algebra in dimension three. Starting dimension four there are partial classifications. The list of isomorphisms classes of four-dimensional nilpotent Leibniz algebras has been given by Albeverio et al. [**8**]. The papers [**22**], [**23**], [**24**], [**40**] and [**41**] are devoted to the classification problem of low-dimensional complex solvable Leibniz algebras. In dimensions 5–10 there are classifications of filiform parts of nilpotent Leibniz algebras. The notion of filiform Leibniz algebra was introduced by Ayupov and Omirov [**11**]. According to Ayupov-Gómez-Omirov theorem, the class of all filiform Leibniz algebras is split into three subclasses which are invariant with respect to the action of the general linear group. One of these classes contains the class of filiform Lie algebras. There is a classification of the class of filiform Lie algebras in small dimensions (Gómez-Khakimdjanov) and there is a classification of filiform Lie algebras admitting a nontrivial Malcev Torus (Goze-Khakimdjanov) [**36**]. The other two of the three classes come out from naturally graded non-Lie filiform Leibniz algebras. For this case the isomorphism criteria have been given (see [**34**]). In [**55**] a method of classification of filiform Leibniz algebras based on algebraic invariants has been developed. Then the method has been implemented to low-dimensional cases in [**65**] and [**66**]. The third class that comes out from naturally graded filiform Lie algebras, has been treated in the paper [**52**]. Then the classifications of some subclasses and low-dimensional cases of this class have been given [**57**], [**58**] and [**59**].

DEFINITION 3.15. *An action of algebraic group G on a variety Z is a morphism $\sigma : G \times Z \longrightarrow Z$ with*
(i) $\sigma(e, z) = z$*, where e is the unit element of G and $z \in Z$,s*
(ii) $\sigma(g, \sigma(h, z)) = \sigma(gh, z)$*, for any $g, h \in G$ and $z \in Z$.*

We shortly write gz for $\sigma(g, z)$, and call Z a G-variety.

DEFINITION 3.16. *A morphism $f : Z \longrightarrow K$ is said to be invariant if $f(gz) = f(z)$ for any $g \in G$ and $z \in Z$.*

Let V be a vector space of dimension n over an algebraically closed field K (char$K$=0). Bilinear maps $V \times V \to V$ form a vector space $Hom(V \otimes V, V)$ of dimension n^3, which can be considered together with its natural structure of an affine algebraic variety over K and denoted by $Alg_n(K) \cong K^{n^3}$. An n-dimensional algebra L over K can be regarded as an element $\lambda(L)$ of $Alg_n(K)$ via the bilinear mapping $\lambda : L \otimes L \to L$ defining a binary algebraic operation on L : let $\{e_1, e_2, \dots, e_n\}$ be a basis of the algebra L. Then the table of multiplication of L is represented by point (γ_{ij}^k) of this affine space as follows:

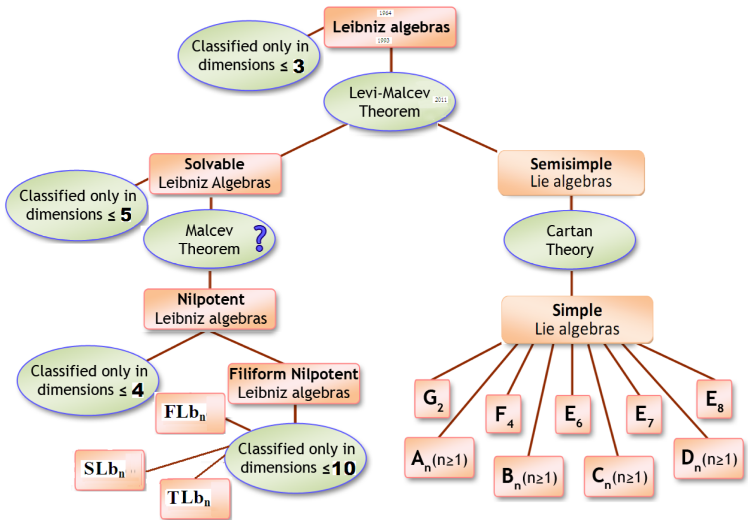
$$\lambda(e_i, e_j) = \sum_{k=1}^{n} \gamma_{ij}^k e_k.$$

Here γ_{ij}^k are called *structure constants* of L. The linear reductive group $GL_n(K)$ acts on $Alg_n(K)$ by $(g * \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y)))$("transport of structure"). Two algebras λ_1 and λ_2 are isomorphic if and only if they belong to the same orbit under this action.

Let $LB_n(K)$ be a subvariety of $Alg_n(K)$ consisting of all n-dimensional Leibniz algebras over a field K. It is stable under the above mentioned action of $GL_n(K)$. As a subset of $Alg_n(K)$ the set $LB_n(K)$ is specified by the system of equations with respect to structure constants γ_{ij}^k:

$$\sum_{l=1}^{n} (\gamma_{jk}^l \gamma_{il}^m - \gamma_{ij}^l \gamma_{lk}^m + \gamma_{ik}^l \gamma_{lj}^m) = 0, \quad \text{where} \quad i, j, k = 1, 2, ..., n.$$

The first naive way to describe $LB_n(K)$ is to solve this quadratic system of equations with respect to γ_{ij}^k, which is somewhat cumbersome. It has been done for some classes of algebras in low-dimensional cases [**12**], [**39**], etc. The complexity of the computations increases much with increasing of the dimension. Therefore one has to create some appropriate methods of study. However, to classify whole $LB_n(K)$ for a fixed large n is a hopeless task. Hence one considers some subclasses of $LB_n(K)$ to be classified. We propose a structure scheme for $LB_n(\mathbb{C})$ which is counterpart of the structural scheme of finite dimensional complex Lie algebras.



Picture 2. Structural scheme for Leibniz algebras.

In [**11**] the authors split the class of complex filiform Leibniz algebras obtained from naturally graded filiform non-Lie Leibniz algebras into two disjoint classes. If we add here the class of filiform Leibniz algebras appearing from naturally graded filiform Lie algebras then the final result can be written as follows.

THEOREM 3.1. *Any $(n+1)$-dimensional complex non-Lie filiform Leibniz algebra L admits a basis $\{e_0, e_1, e_2, ..., e_n\}$ such that L has a table of multiplication one of the following form (unwritten product are supposed to be zero)*

$$FLb_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1 \\ [e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + ... + \alpha_{n-1} e_{n-1} + \theta e_n, & \\ [e_j, e_1] = \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + ... + \alpha_{n+1-j} e_n, & 1 \leq j \leq n-2 \end{cases}$$

$$SLb_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 2 \leq i \leq n-1 \\ [e_0, e_1] = \beta_3 e_3 + \beta_4 e_4 + ... + \beta_n e_n, & \\ [e_1, e_1] = \gamma e_n, & \\ [e_j, e_1] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + ... + \beta_{n+1-j} e_n, & 2 \leq j \leq n-2 \end{cases}$$

$$TLb_{n+1} = \begin{cases} [e_0, e_0] = e_n, & \\ [e_1, e_1] = \alpha e_n, & \\ [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1 \\ [e_0, e_1] = -e_2 + \beta e_n, & \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le n-1 \\ [e_i, e_j] = -[e_j, e_i] \in lin < e_{i+j+1}, e_{i+j+2}, \dots, e_n >, & 1 \le i \le n-3, \\ & 2 \le j \le n-1-i \\ [e_{n-i}, e_i] = -[e_i, e_{n-i}] = (-1)^i \delta e_n, & 1 \le i \le n-1 \end{cases}$$

where $[\cdot,\cdot]$ *is the multiplication in* L *and* $\delta \in \{0, 1\}$ *for odd* n *and* $\delta = 0$ *for even* n.

4. Semisimple case

There is one more case which should be mentioned here. As it has been mentioned above the quotient of a Leibniz algebra with respect to the ideal I generated by squares is a Lie algebra and I itself can be regarded as a module over this Lie algebra. There are results on description of such a Leibniz algebras with a fixed quotient Lie algebra. The case $L/I = sl_2$ has been treated in [**53**]. In [**19**] the authors describe Leibniz algebras L with $L/I = sl_2 \dotplus R$, where R is solvable and $dimR = 2$. When $L/I = sl_2 \dotplus R$ with $dimR = 3$ the result has been given in [**64**]. All these results are based on the classical result on description of irreducible representations of the simple Lie algebra sl_2. Unfortunately, the decomposition of a semisimple Leibniz algebra into direct sum of simple ideals is not true. Here an example from [**20**] supporting this claim. Let L be a complex Leibniz algebra satisfying the following conditions

(a) $L/I \cong sl_2^1 \oplus sl_2^2$;
(b) $I = I_{1,1} \oplus I_{1,2}$ such that $I_{1,1}, I_{1,2}$ are irreducible sl_2^1-modules and $dimI_{1,1} = dimI_{1,2}$;
(c) $I = I_{2,1} \oplus I_{2,2} \oplus ... \oplus I_{2,m+1}$ such that $I_{2,k}$ are irreducible sl_2^2-modules with $1 \le k \le m+1$.

Then there is a basis $\{e_1, f_1, h_1, e_2, f_2, h_2, x_0^1, x_1^1, x_2^1, ..., x_m^1, x_0^2, x_1^2, x_2^2, ..., x_m^2\}$ such that the table of multiplication of L in this basis is represented as follows:

$$L \cong \begin{cases} [e_i, h_i] = -[h_i, e_i] = 2e_i, & \\ [e_i, f_i] = -[f_i, e_i] = h_i, & \\ [h_i, f_i] = -[f_i, h_i] = 2f_i, & \\ [x_k^i, h_1] = (m-2k)x_k^i, & 0 \le k \le m, \\ [x_k^i, f_1] = x_{k+1}^i, & 0 \le k \le m-1, \\ [x_k^i, e_1] = -k(m+1-k)x_{k-1}^i, & 1 \le k \le m, \\ [x_j^1, e_2] = [x_j^2, h_2] = x_j^2, & \\ [x_j^1, h_2] = [x_j^2, f_2] = -x_j^1, & \end{cases}$$

with $1 \le i \le 2$ and $0 \le j \le m$. The algebra L can not be represented as a direct sum of simple Leibniz algebras.

5. Generalizations

Several generalizations of Leibniz algebras have been introduced and studied. We list just few of them below.

- n-Leibniz algebras have been introduced in [**27**]

- The papers [**7**], [**18**] contain results on Cartan subalgebras, nilpotency properties of n-Leibniz algebras (also see [**54**]).
- Results on an enveloping algebra and PBW theorem for n-Leibniz algebras are given in [**25**]).
- Leibniz superalgebras are introduced and studied in [**10**].

6. Approaches applied: Classification of complex filiform Leibniz algebras

In the classification problem of algebraic structures the isomorphism invariants play an important role. The main isomorphism invariants have been used to classify and distinguish classes of algebras are given as follows.

6.1. Discrete Invariants.

- **The dimension of characteristic ideals and the nilindex**;
- **The characteristic sequence**;
- **The rank of nilpotent Lie algebras**;
- **The dimension of group (co)homologies**;
- **The characteristic of the derivation algebra**;
- **The dimension of the center, the right and left annihilators**;
- **The Dixmier Invariant**.

6.2. Algebraic Invariants: new.

6.2.1. *Vector space of algebras.* Let n be an nonnegative integer. A solution to the classification problem for n-dimensional nonassociative algebras consists in setting up a list of examples which represents each isomorphism class exactly once. Such a list may also be interpreted as a parametrization of the orbit space $GL(V)\diagdown Hom(V \bigotimes V, V)$, where V is an n-dimensional vector space acted upon canonically by the general linear group, with the induced diagonal action on $V \bigotimes V$ and its natural extension to $Hom(V \bigotimes V, V)$. In this way, the classification problem for n-dimensional algebras relates to questions in invariant theory.

6.2.2. *Group action.* A Leibniz algebra on n-dimensional vector space V over a field K can be regarded as a pair $L = (V, \lambda)$, where λ is a Leibniz algebra law on V, the underlying vector space to L. As above by $LB_n(K)$ we denote the set of all Leibniz algebra structures on the vectors space V over K. It is a subspace of the linear space of all bilinear mappings $V \times V \longrightarrow V$.

The linear reductive group $GL_n(K)$ acts on $Alg_n(K)$ by $(g * \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y)))$ ("transport of structure").

Definition 6.1. *Two laws λ_1 and λ_2 from $LB_n(K)$ are said to be isomorphic, if there is $g \in GL_n(K)$ such that*

$$\lambda_2(x, y) = (g * \lambda_1)(x, y) = g^{-1}(\lambda_1(g(x), g(y)))$$

for all $x, y \in V$. Thus we get an action of $GL_n(K)$ on $LB_n(K)$.

The set of the laws isomorphic to λ is called the orbit of λ.

Remind that $Lb_n(K)$ denote the variety of all filiform Leibniz algebra structures on n-dimensional vector space V over a field K.

6.2.3. *Strategy.* Our strategy to classify $Lb_n(\mathbb{C})$ is as follows:

(1) We break up the class $Lb_n(\mathbb{C})$ into three subclasses. They are denoted by FLb_n, SLb_n and TLb_n, respectively. Two of the classes come out from naturally graded non-Lie filiform Leibniz algebras and the third one comes out from naturally graded filiform Lie algebras. Note that filiform Lie algebras are in TLb_n.
(2) We choose bases called adapted and write each of FLb_n, SLb_n and TLb_n in terms of their structure constants.
(3) Consider respective subgroup G_{ad} of $GL_n(\mathbb{C})$ operating on FLb_n, SLb_n and TLb_n. This subgroup is called adapted transformations group. Hence the classification problem reduces to the problem of classifying the orbits of G_{ad} acting on $Lb_n(\mathbb{C})$.
(4) Define elementary base change. We show that only few types of elementary transformations act on $Lb_n(\mathbb{C})$. Therefore, it suffices to consider the only specified base changes.

6.2.4. *Results.*

(1) The general isomorphism criteria for each of FLb_n, SLb_n and TLb_n are given by using rational invariant functions depending on structure constants of algebras (see[**52**] and [**55**]).
(2) The classes FLb_n, SLb_n and TLb_n are classified for $n \leq 10$ (see [**28**], [**56**] for SLb_9, [**65**], [**66**] for FLb_n and SLb_n, $n = 5, 6, 7$, [**71**] for FLb_{10}, [**48**] for SLb_{10} and [**1**], [**2**], [**38**], [**57**] for TLb_n, $n = 5 - 10$). Each of FLb_n, SLb_n and TLb_n is broken down into disjoint invariant, with respect to base change, subsets and for each of the subsets the respective set of rational invariants (orbit functions) are given.
(3) In [**58**] and [**59**] some subclasses of TLb_n are represented as a Leibniz central extensions of a Lie algebra and they are classified up to isomorphisms.

6.3. (Co)homological approach. Let L be a Leibniz algebra and V be a vector space over a field K ($Char\ K \neq 2$. Then a bilinear map $\theta : L \times L \longrightarrow V$ with the property

$$\theta(x, [y, z]) = \theta([x, y], z) - \theta([x, z], y), \quad \text{for all} \quad x, y, z \in L$$

is called Leibniz cocycle. The set of all Leibniz cocycles is denoted by $ZL^2(L, V)$. Let $\theta \in ZL^2(L, V)$. Then, we set $L_\theta = L \oplus V$ and define a bracket $[\cdot, \cdot]$ on L_θ by

$$[x + v, y + w] = [x, y]_L + \theta(x, y),$$

where $[\cdot, \cdot]_L$ is the bracket on L.

The proof of the following lemma can be found by a simple computation.

LEMMA 6.1. *L_θ is a Leibniz algebra if and only if θ is a Leibniz cocycle.*

The Leibniz algebra L_θ is called a **central extension** of L by V. Let $\nu : L \longrightarrow V$ be a linear map, and define $\eta(x, y) = \nu([x, y])$. Then it is easy to see that η is a Leibniz cocycle called **coboundary**. The set of all coboundaries is denoted by $BL^2(L, V)$. Clearly, $BL^2(L, V)$ is a subgroup of $ZL^2(L, V)$. We call the factor space, denoted by $HL^2(L, V) = ZL^2(L, V)/BL^2(L, V)$, the **second cohomology** group of L by V.

The following lemma shows that the central extension of a given Leibniz algebra L is limited to the coboundary level.

LEMMA 6.2. *Let L be a Leibniz algebra and η be a coboundary, then the central extensions L_θ and $L_{\theta+\eta}$ are isomorphic.*

When constructing a central extension of a Leibniz algebra L as $L_\theta = L \oplus V$, we want to restrict θ such a way that the center of L_θ equals V. In this way, we discard constructing the same Leibniz algebra as central extension of different Leibniz algebras.

The center of a Leibniz algebra L is defined as follows:

$$C(L) = \{x \in L \mid \ [x, L] = [L, x] = 0\}.$$

For $\theta \in ZL^2(L, V)$ set

$$\theta^\perp = \{x \in L \mid \theta(x, L) = \theta(L, x) = 0\},$$

which is called the radical of θ (Rad(θ)=$\theta^\perp$). We conclude that any Leibniz algebra with a nontrivial center can be obtained as a central extension of a Leibniz algebra of smaller dimension.

The proof of the following lemma is straightforward.

LEMMA 6.3. *If $\theta \in ZL^2(L, V)$ then $C(L_\theta) = (\theta^\perp \cap C(L)) + V$.*

As a consequence of this lemma we get the following criterion.

COROLLARY 6.1. *$\theta^\perp \cap C(L) = \{0\}$ if and only if $C(L_\theta) = V$.*

Let $e_1, ..., e_k$ be a basis of V and $\theta \in ZL^2(L, V)$. Then

$$\theta(x, y) = \sum_{i=1}^{k} \theta_i(x, y)e_i,$$

where $\theta_i \in ZL^2(L, K)$. Furthermore, θ is a coboundary if and only if all θ_i are.

The automorphism group $Aut(L)$ acts on $ZL^2(L, V)$ by $\phi\theta(x, y) = \theta(\phi(x), \phi(y))$ and $\eta \in BL^2(L, V)$ if and only if $\phi\eta \in BL^2(L, V)$. This induces an action of $Aut(L)$ on $HL^2(L, V)$. The proof of the following theorem can be carried out for Leibniz algebras by a minor modification of that for Lie algebras.

THEOREM 6.1. *Let $\theta(x, y) = \sum\limits_{i=1}^{k} \theta_i(x, y) e_i$ and $\eta(x, y) = \sum\limits_{i=1}^{k} \eta_i(x, y) e_i$ be two elements of $HL^2(L, V)$. Suppose that $\theta^\perp \cap C(L) = \eta^\perp \cap C(L) = \{0\}$. Then $L_\theta \cong L_\eta$ if and only if there is a $\varphi \in Aut(L)$ such that $\varphi\eta_i$ span the same subspace of $HL^2(L, V)$ as θ_i.*

Let $L = I_1 \oplus I_2$, where I_1 and I_2, are ideals of L. Suppose that I_2 is contained in the center of L. Then I_2 is called a central component of L. In order to keep away from the Leibniz algebras with central components we use the following criterion.

LEMMA 6.4. *Let $\theta(x, y) = \sum\limits_{i=1}^{k} \theta_i(x, y) e_i \in HL^2(L, V)$ be such that $\theta^\perp \cap C(L) = \{0\}$. Then L_θ has no central components if and only if $\theta_1, ..., \theta_k$ are linearly independent in $HL^2(L, K)$.*

Let $G_k(HL^2(L, K))$ be the Grassmanian of subspaces of dimension k in $HL^2(L, K)$. One makes $Aut(L)$ act on $G_k(HL^2(L, K))$ as follows: $W = < \vartheta_1, \vartheta_2, ..., \vartheta_k > \in G_k(HL^2(L, K))$,

$$\phi W = < \phi\vartheta_1, \phi\vartheta_2, ..., \phi\vartheta_k > .$$

This definition is legitimate because if $\{\vartheta_1, \vartheta_2, ..., \vartheta_k\}$ is linear independent so is $\{\phi\vartheta_1, \phi\vartheta_2, ..., \phi\vartheta_k\}$.

Define

$$U_k(L) = \left\{W =< \vartheta_1, \vartheta_2, ..., \vartheta_k >\in G_k(HL^2(L,K)) : \vartheta_i^{\perp} \cap Z(L) = \{0\}, i = 1, 2, ..., k\right\}.$$

LEMMA 6.5. *The set $U_k(L)$ is stable under the action of $Aut(L)$.*

The set of orbits under the action $Aut(L)$ on $U_k(L)$ is denoted by $U_k(L)/Aut(L)$. Here is an analogue of Skejelbred-Sund theorem (see [**72**]) for Leibniz algebras.

THEOREM 6.2. *There exists a canonical one-to-one map from $U_k(L)/Aut(L)$ onto the set of isomorphism classes of Leibniz algebras without direct abelian factor which are central extensions of L by K^k and have $k-$dimensional center.*

6.3.1. *The classification procedure.* This section deals with the procedure to construct nilpotent Leibniz algebras which fixed dimension given that in low-dimensions.

Let a nilpotent Leibniz algebra E over a field K of dimension $n-k$ is given as input. The outputs of the procedure are all nilpotent Leibniz algebras L of dimension n such that $L/C(L) \cong E$, and L has no central components. It runs as follows.

(1) For a given algebra of smaller dimension, we list at first its center (or the generators of its center), to help us identify the 2−cocycles satisfying $\theta^{\perp} \cap C(E) = 0$.
(2) We also list its derived algebra (or the generators of the derived algebra), which is needed in computing the coboundaries $BL^2(E,K)$.
(3) Then we compute all the 2−cocycles $ZL^2(E,K)$ and $BL^2(E,K)$ and compute the set $HL^2(E,K)$ of cosets of $BL^2(E,K)$ in $ZL^2(E,K)$. For each fixed algebra E with given base $\{e_1, e_2, ..., e_k\}$, we may represent a 2−cocycles θ by a matrix $\theta = \sum_{i,j=1}^{k} c_{ij}\Delta_{ij}$, where Δ_{ij} is the $k \times k$ matrix with (i,j) element being 1 and all the others 0. When computing the 2−cocycles, we will just list all the constraints on the elements c_{ij} of the matrix θ.
(4) We have $ZL^2(L,K) = BL^2(L,K) \oplus W$, where W is a subspace of $ZL^2(E,K)$, complementary to $BL^2(E,K)$, and

$$BL^2(E,K) = \{df \mid f \in C^1(E,K) = E^*,\}$$

where d is the coboundary operator. One easy way to obtain W is as follows. When a nilpotent Leibniz algebra L of dimension $n = r + k$ has a basis in the form $\{e_1, ..., e_r, e_{r+1}, ..., e_{r+k}\}$, where $\{e_1, ..., e_r\}$ are the generators, and $\{e_{r+1}, ..., e_{r+k}\}$ forms a basis for the derived algebra $[L, L]$, with $e_{r+t} = [e_{i_t}, e_{j_t}]$, where $1 \le i_t, j_t < r + t$ and $1 \le t \le k$. Consider $C^1(E,K) = E^*$ generated by the dual basis

$$< f_1, ..., f_r, g_1, ..., g_k >$$

of

$$< e_1, ..., e_r, e_{r+1}, ..., e_{r+k} > .$$

Then

$$BL^2(E,K) = \{dh \mid h \in L^*\} =< df_1, ..., df_r, dg_1, ..., dg_k > .$$

Since $df_i(x,y) = -f_i([x,y]) = 0$, we have $BL^2(E,K) =< dg_1, ..., dg_k >$. So one has

$$ZL^2(E,K) =< dg_1, ..., dg_k > \oplus W.$$

For $\theta \in W$, we may assume that $\theta(e_{i_t}, e_{j_t}) = 0$, $t = 1, ..., k$, otherwise, if $\theta(e_{i_t}, e_{j_t}) = u_{i_t j_t} \neq 0$, we choose $\theta + u_{i_t j_t} dg_t$ instead. When we carry out the group action on W, we do it as if it were done in $HL^2(E,K)$, and may identify $HL^2(E,K)$ with W, by calling all the nonzero elements in W the normalized $2-$cocycles.

(5) Consider $\theta \in HL^2(E,V)$ with $\theta(x,y) = \sum_{i=1}^{k} \theta_i(x,y)e_i$ where $\theta_i \in HL^2(E,K)$ are linearly independent, and $\theta^{\perp} \cap C(E) = 0$.
(6) Find a (maybe redundant) list of representatives of the orbits of $Aut(L)$ acting on the θ from 5.
(7) For each θ found, construct $L = E_{\theta}$. Discard the isomorphic ones (see [**60**] and [**61**]).

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