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ON ISOMORPHISMS AND INVARIANTS OF FINITE DIMENSIONAL COMPLEX FILIFORM LEIBNIZ ALGEBRAS

I. S. Rakhimov¹ and U. D. Bekbaev²
¹Institute for Mathematical Research (INSPEM) and Department of Mathematics, Universiti Putra Malaysia, Malaysia
²Department of Mathematical and Natural Sciences, Turin Polytechnic University in Tashkent, Uzbekistan

In this article, we propose an approach classifying a class of filiform Leibniz algebras. The approach is based on algebraic invariants. The method allows to classify all filiform Leibniz algebras (including filiform Lie algebras) in a given fixed dimensional case.

Key Words: Filiform Leibniz algebra; Isomorphism criterion; Lie algebra; Natural gradation.

2000 Mathematics Subject Classification: Primary 17A32, 17A60, 17B30, 17B70; Secondary 13A50.

1. INTRODUCTION

It is well known that the natural gradation of nilpotent Lie and Leibniz algebras is very helpful in investigation of their structural properties. This technique is more effective when the length of the natural gradation is sufficiently large. In the case when it is maximal, the algebra is called filiform. For applications of this technique, for instance, see Vergne [18], Goze et al. [10] (for Lie algebras) and Ayupov et al. [2] (for Leibniz algebras) case.

The present article deals with a class of nonassociative algebras that generalizes the class of Lie algebras. These algebras satisfy certain identities that were suggested by Loday [12] and Cuvier [5]. When one uses the tensor product instead of external product in the definition of the nth cochain, in order to prove the differential property, that is defined on cochains, it suffices to replace the anticommutativity and Jacobi identity by the Leibniz identity. This is one of the motivations to appear for this class of algebras. It turned out later that they appeared to be related in a natural way to several topics such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, quantum physics etc., as a generalization of the corresponding applications of Lie algebras to these topics.

The (co)homology theory, representations, and related problems of Leibniz algebras were studied by Cuvier [4], Loday et al. [14], Liu et al. [15], and others. A good survey about these all and related problems is Loday et al. [13].

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Address correspondence to I. S. Rakhimov, Institute for Mathematical Research (INSPEM), Department of Mathematics, FS, UPM 43400, Serdang, Selangor, Kuala Lumpur, Malaysia; E-mail: risamiddin@mail.ru

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The problems related to the group theoretical realizations of Leibniz algebras are studied by Kinyon et al. [11] and others. Deformation theory of Leibniz algebras and related physical applications of it are initiated by Fialowski et al. [7]. Problems concerning Cartan subalgebras, solvability, and weight spaces were studied by Albeverio et al. [1] and Omirov [16]. The notion of simple Leibniz algebra was suggested by Dzhumadil’daev et al. [6], who obtained some results concerning special cases of simple Leibniz algebras. The article is organized as follows. Section 2 collects basic definitions, notations, and conventions used in the article. Section 3 is devoted to the adapted basis and the adapted transformations. Here we mention an isomorphism criterion from Gómez et al. [8] for filiform Leibniz algebras whose naturally gradation is non-Lie filiform Leibniz algebra. Then we rewrite it adjusting to our purpose. The main results of the article are in Sections 4 and 5. In Section 4 we give an algorithm for algebraic classification of finite dimensional complex filiform Leibniz algebras derived from the naturally graded non-Lie filiform Leibniz algebra in terms of invariant functions (Sections 4.1 and 4.2). Section 4.3 deals with the class of filiform Leibniz algebras whose natural gradation is a filiform Lie algebra. Here we simplify the table of multiplication and keep track of the behavior of the structure constants under the adapted base change. Section 5 contains implementations of the results in some low dimensional cases.

2. PRELIMINARIES

Let $V$ be a vector space of dimension $n$ over an algebraically closed field $K$ (char $K=0$). Bilinear maps $V \times V \rightarrow V$ form a vector space $\text{Hom}(V \otimes V, V)$ of dimensional $n^3$, which can be considered together with its natural structure of an affine algebraic variety over $K$ and denoted by $\text{Alg}_n(K) \cong K^{n^3}$. An $n$-dimensional algebra $L$ over $K$ can be considered as an element $\lambda(L)$ of $\text{Alg}_n(K)$ via the bilinear mapping $\lambda : L \otimes L \rightarrow L$ defining a binary algebraic operation on $L$: let \{e_1, e_2, \ldots, e_n\} be a basis of the algebra $L$. Then the table of multiplication of $L$ is represented by point $(\lambda_{ij}^k)$ of this affine space as follows:

$$\lambda(e_i, e_j) = \sum_{k=1}^{n} \lambda_{ij}^k e_k.$$

Here $\lambda_{ij}^k$ are called structure constants of $L$. The linear reductive group $GL_n(K)$ acts on $\text{Alg}_n(K)$ by $(g \ast \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y)))$ (“transport of structure”). Two algebras $\lambda_1$ and $\lambda_2$ are isomorphic if and only if they belong to the same orbit under this action. Recall that an algebra $L$ over a field $F$ is called a Leibniz algebra if it satisfies the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

where $[\cdot, \cdot]$ denotes the multiplication in $L$. A skew-symmetric Leibniz algebra is a Lie algebra. In this case, (1) is just the Jacobi identity.
Let $LB_n(K)$ be a subvariety of $Alg_n(K)$ consisting of all $n$-dimensional Leibniz algebras over $K$. It is stable under the above mentioned action of $GL_n(K)$. As a subset of $Alg_n(K)$ the set $LB_n(K)$ is specified by the system of equations with respect to structure constants $\gamma_{ij}^k$

$$\sum_{l=1}^n (\gamma_{jk}^l \gamma_{il}^m - \gamma_{ij}^l \gamma_{lk}^m + \gamma_{il}^j \gamma_{jk}^m) = 0, \quad \text{where } i, j, k, m = 1, 2, \ldots, n.$$ 

The first naive way to describe $LB_n(K)$ is to solve this quadratic system with respect to $\gamma_{ij}^k$, which is somewhat cumbersome. It has been done for low-dimensional Leibniz algebras ($n \leq 3$). The complexity of the computations increases much with increasing of dimension. Therefore, usually one has to create some appropriate methods of investigation. However, to classify whole $LB_n(K)$ for any fixed $n$ is a hopeless task. Hence one considers some subclasses of $LB_n(K)$ to be classified.

Let $L$ be a Leibniz algebra. We define the lower central series

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1.$$ 

**Definition 2.1.** A Leibniz algebra $L$ is called nilpotent if there exists $s \in \mathbb{N}$ such that $L^s = 0$.

**Definition 2.2.** A Leibniz algebra $L$ is said to be filiform if $\dim L^i(L) = n - i$, where $n = \dim L$ and $2 \leq i \leq n$.

Let $L^s_n$ denote the class of all $n$-dimensional filiform Leibniz algebras. Clearly, it is a subclass of nilpotent Leibniz algebras. Let $L$ be a nilpotent Leibniz algebra with nilindex $s$. Consider $L_i = L^i/L^{i+1}$, $1 \leq i \leq s - 1$, and $grL = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1}$. Then $[L_i, L_j] \subseteq L_{i+j}$, and we obtain the graded algebra $grL$.

**Definition 2.3.** If a Leibniz algebra $L'$ is isomorphic to a filiform naturally graded algebra $grL$, then $L'$ is said to be naturally graded filiform Leibniz algebra.

Later on all algebras are supposed to be over the field of complex numbers $\mathbb{C}$ and omitted products of basis vectors are supposed to be zero. The following theorem summarizes the results of Ayupov et al. [2] and Vergne [18].

**Theorem 2.1.** Any complex $(n+1)$-dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise non-isomorphic algebras:

$$NGF_1 = \begin{cases} [e_0, e_0] = e_2, \\
[e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \end{cases}$$

$$NGF_2 = \begin{cases} [e_0, e_0] = e_2, \\
[e_i, e_0] = e_{i+1}, & 2 \leq i \leq n-1, \end{cases}$$

$$NGF_3 = \begin{cases} [e_i, e_0] = -[e_0, e_i] = e_{i+1}, & 1 \leq i \leq n-1, \\
[e_i, e_{-n}] = -[e_{-n}, e_i] = x(-1)^{i+1}e_n, & 1 \leq i \leq n-1, \\
x \in \{0, 1\} \text{ for odd } n \text{ and } x = 0 \text{ for even } n. \end{cases}$$
It is clear that $NGF_3$ is a Lie algebra, but neither $NGF_1$ nor $NGF_2$ is the case.

Based on this theorem, the class of fixed dimensional filiform Leibniz algebras can be split into three disjoint classes as follows (see Gómez et al. [8]).

**Theorem 2.2.** Any $(n + 1)$-dimensional complex filiform Leibniz algebra admits a basis $\{e_0, e_1, \ldots, e_n\}$ called adapted, such that the table of multiplication of the algebra has one of the following forms:

\[
\begin{align*}
FLb_{n+1} = & \begin{cases} 
[e_0, e_0] = e_2, \\
[e_i, e_0] = e_{i+1}, & 1 \leq i \leq n - 1, \\
[e_0, e_1] = x_3 e_3 + x_4 e_4 + \cdots + x_{n-1} e_{n-1} + \theta e_n, \\
[e_j, e_1] = x_3 e_{j+2} + x_4 e_{j+3} + \cdots + x_{n+1-j} e_n, & 1 \leq j \leq n - 2,
\end{cases} \\
SLb_{n+1} = & \begin{cases} 
[e_0, e_0] = e_2, \\
[e_i, e_0] = e_{i+1}, & 2 \leq i \leq n - 1, \\
[e_0, e_1] = \beta_3 e_3 + \beta_4 e_4 + \cdots + \beta_n e_n, \\
[e_j, e_1] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + \cdots + \beta_{n+1-j} e_n, & 2 \leq j \leq n - 2,
\end{cases} \\
TLb_{n+1} = & \begin{cases} 
[e_0, e_0] = e_{i+1}, & 1 \leq i \leq n - 1, \\
[e_i, e_i] = -e_{i+1}, & 2 \leq i \leq n - 1, \\
[e_0, e_0] = b_{0,0} e_n, \\
[e_0, e_1] = -e_2 + b_{0,1} e_n, \\
[e_i, e_1] = b_{1,1} e_n, \\
[e_j, e_j] = -[e_j, e_i] \in \text{span}_{C}[e_{i+j+1}, e_{i+j+2}, \ldots, e_n] \mid 1 \leq i \leq n - 3, \\
& 2 \leq j \leq n - 1 - i,
\end{cases}
\end{align*}
\]

where $a_{i,j}^k$, $b_{i,j} \in C$ and $b_{i,n-i} = b$ whenever $1 \leq i \leq n - 1$, and $b = 0$ for even $n$.

The above theorem means that the natural gradation of a filiform Leibniz algebra may be an algebra from one of $NGF_i$ for $i = 1, 2, 3$.

3. ADAPTED BASE CHANGE AND ISOMORPHISM CRITERIA

In this section we simplify the isomorphic action of $GL_n$ (“transport of structure”) on the class of algebras coming out from the naturally graded non-Lie filiform Leibniz algebras. The details of the proofs can be found in Gómez et al. [8].

Let $L$ be a filiform Leibniz algebra defined on a vector space $V$ and $\{e_0, e_1, \ldots, e_n\}$ be an adapted basis of $L$. 
The closed subgroup of $GL(V)$ consisting of all linear transformation sending one adapted basis to another is said to be adapted for the structure of $L$. This subgroup is denoted by $G_{ad}$. In $G_{ad}$, we consider the following isomorphisms, called elementary:

first type $- \tau(a, b, k) = \begin{cases} f(e_0) = e_0 + ae_1, \\ f(e_i) = e_1 + be_1, \\ f(e_{i+1}) = [f(e_i), f(e_0)], \quad 1 \leq i \leq n - 1, 2 \leq k \leq n, \\ f(e_2) = [f(e_0), f(e_0)]. \end{cases}$

second type $- \theta(a, b) = \begin{cases} f(e_0) = ae_0 + be_1, \\ f(e_i) = (a + b)e_1 + b(\theta - z_n)e_{n-1}, \quad a(a + b) \neq 0, \\ f(e_{i+1}) = [f(e_i), f(e_0)], \quad 1 \leq i \leq n - 1, \\ f(e_2) = [f(e_0), f(e_0)]. \end{cases}$

third type $- \sigma(b, n) = \begin{cases} f(e_0) = e_0, \\ f(e_i) = e_1 + be_n, \\ f(e_{i+1}) = [f(e_i), f(e_0)], \quad 2 \leq i \leq n - 1, \\ f(e_2) = [f(e_0), f(e_0)]. \end{cases}$

fourth type $- \eta(a, k) = \begin{cases} f(e_0) = e_0 + ae_1, \\ f(e_i) = e_1, \\ f(e_{i+1}) = [f(e_i), f(e_0)], \quad 2 \leq i \leq n - 1, 2 \leq k \leq n, \\ f(e_2) = [f(e_0), f(e_0)]. \end{cases}$

fifth type $- \delta(a, b, d) = \begin{cases} f(e_0) = ae_0 + be_1, \\ f(e_i) = de_1 - \frac{bd}{a} e_{n-1}, \quad ad \neq 0, \\ f(e_{i+1}) = [f(e_i), f(e_0)], \quad 2 \leq i \leq n - 1, \\ f(e_2) = [f(e_0), f(e_0)]. \end{cases}$

where $a, b, d \in \mathbb{C}$.

**Proposition 3.1.**

(a) Let $f$ be an adapted transformation of $FLb_{n+1}$; then

$$f = \tau(a_n, b_n, n) \circ \tau(a_{n-1}, a_{n-1}, n - 1) \circ \cdots \circ \tau(a_2, a_2, 2) \circ \theta(a_0, a_1).$$

(b) Let $f$ be an adapted transformation of $SLb_{n+1}$; then

$$f = \sigma(b_n, n) \circ \eta(a_n, n) \circ \eta(a_{n-1}, a_{n-1}, n - 1) \circ \cdots \circ \eta(a_2, a_2) \circ \delta(a_0, a_1, b_1).$$
Proposition 3.2.
(a) The transformation

\[ \tau(a_n, b_n, n) \circ \tau(a_{n-1}, a_{n-1}, n-1) \circ \cdots \circ \tau(a_2, a_2, 2) \]

preserves the structure constants of algebras from \( FLb_{n+1} \).

(b) The transformation

\[ \sigma(b_n, n) \circ \eta(a_n, n) \circ \eta(a_{n-1}, n-1) \circ \cdots \circ \eta(a_2, 2) \]

preserves the structure constants of algebras from \( SLb_{n+1} \).

Thus the action of \( GL_{ad}(V) \) on \( FLb_{n+1} \) and \( SLb_{n+1} \) can be reduced to the action of the elementary transformations of the second and the fifth types, respectively.

The next two theorems are reformulations of the corresponding results of Gómez et al. [8] on isomorphism criteria for filiform Liebniz algebras appearing from the naturally graded non Lie filiform Liebniz algebras.

Introduce the following series of functions:

\[ \varphi_i(y; z) = \varphi_i(y, \bar{z}_3, \bar{z}_4, \ldots, \bar{z}_n, \bar{z}_{n+1}) \]

\[ = (1 + y)z_i - \sum_{k=3}^{t-1} \left( \frac{k-1}{k-2} \right) y^2 z_{i+k-4} + \left( \frac{k-1}{k-3} \right) y^3 \sum_{i_j=k+2}^{t-1} z_{i+3-i_j} \bar{z}_{i+1-k} \]

\[ + \left( \frac{k-1}{k-4} \right) y^3 \sum_{i_j=k+3}^{t-1} \sum_{i_l} z_{i+3-i_j} z_{i+3-i_l} \bar{z}_{i+1-k} + \cdots \]

\[ = + \left( \frac{k-1}{1} \right) y^{k-1} \sum_{i_j=k+3}^{t-1} \sum_{i_l} z_{i+3-i_j-k} z_{i+1-k+5} \]

\[ + y^{k-1} \sum_{i_j=k+3}^{t-1} \sum_{i_l} z_{i+3-i_j-k} z_{i+4+2k} \]

for \( 3 \leq t \leq n \).

Theorem 3.1. Two algebras \( L(\alpha) \) and \( L(\alpha') \) from \( FLb_{n+1} \), where \( \alpha = (\alpha_3, \alpha_4, \ldots, \alpha_n, 0) \), and \( \alpha' = (\alpha'_3, \alpha'_4, \ldots, \alpha'_n, 0) \) are isomorphic if and only if there exist complex numbers \( A \) and \( B \) such that \( A(A + B) \neq 0 \) and the following conditions hold:

\[ \alpha'_t = \frac{1}{A^{n-t}} \varphi'_t \left( \frac{B}{A}; \alpha \right), \quad 3 \leq t \leq n, \]

\[ \theta' = \frac{1}{A^{n+1}} \varphi_{n+1} \left( \frac{B}{A}; \alpha \right). \]
Let

\[ \psi_t(y; z) = \psi_t(y; z_3, z_4, \ldots, z_n, z_{n+1}) \]

\[ = z_t - \sum_{k=3}^{t-1} \left( \frac{k-1}{k-2} \right) y_{2t-2-k} + \left( \frac{k-1}{k-3} \right) y^2 \sum_{i_1=k+2}^{t} z_{t+1-i_1}z_{i_1+1-k} \]

\[ + C_{k-1}^{t-4} \sum_{i_2=k+3}^{t} \sum_{i_3=k+3}^{t} z_{t+3-i_2}z_{i_2+3-i_1}z_{i_1+1-k} + \cdots \]

\[ + C_{k-1}^{t-2} \sum_{i_2=2k-2}^{t} \sum_{i_3=2k-2}^{t} \cdots \sum_{i_4=2k-2}^{t} z_{t+3-i_2+3-i_3}z_{i_3+3-i_4}z_{i_4+1-k} \]

\[ + y^{k-1} \sum_{i_2=2k-1}^{t} \sum_{i_3=2k-1}^{t} \cdots \sum_{i_4=2k-1}^{t} z_{t+3-i_2+3-i_3}z_{i_3+3-i_4}z_{i_4+1-k} \]

where \( 3 \leq t \leq n \),

and

\[ \psi_{n+1}(y; z) = z_{n+1}. \]

**Theorem 3.2.** Two algebras \( L(\beta) \) and \( L(\beta') \) from \( SL_{n+1} \), where \( \beta = (\beta_1, \beta_2, \ldots, \beta_n, z) \), and \( \beta' = (\beta'_1, \beta'_2, \ldots, \beta'_n, z') \), are isomorphic if and only if there exist complex numbers \( A, B, \) and \( D \) such that \( AD \neq 0 \) and the following conditions hold:

\[ \beta'_t = \frac{1}{A^{t-2}} \frac{D}{A} \psi_t \left( \frac{B}{A} ; \beta \right), \quad 3 \leq t \leq n-1, \]

\[ \beta'_n = \frac{1}{A^{n-2}} \frac{D}{A} \frac{B}{A} \psi_n \left( \frac{B}{A} ; \beta \right), \]

and

\[ z' = \frac{1}{A^{n-2}} \left( \frac{D}{A} \right)^2 \psi_{n+1} \left( \frac{B}{A} ; \beta \right). \]

Now we commence to create the classification procedure for \( L_{n+1} \).

To simplify notation let us agree that in the above case for transition from \( L(\alpha) \) to \( L(\alpha') \), and from \( L(\beta) \) to \( L(\beta') \) we write \( \alpha' = \rho \left( \frac{1}{A} ; \frac{B}{A} ; \alpha \right) \) and \( \beta' = v \left( \frac{1}{A}, \frac{B}{A}, \frac{D}{A} ; \beta \right) \), respectively, where

\[ \rho \left( \frac{1}{A} ; \frac{B}{A} ; \alpha \right) = \left( \rho_1 \left( \frac{1}{A} ; \frac{B}{A} ; \alpha \right), \rho_2 \left( \frac{1}{A} ; \frac{B}{A} ; \alpha \right), \ldots, \rho_{n-1} \left( \frac{1}{A} ; \frac{B}{A} ; \alpha \right) \right), \]

with

\[ \rho_t(x, y, z) = x' \varphi_{t-2}(y, z) \quad \text{for} \ 1 \leq t \leq n-2, \]

\[ \rho_{n-1}(x, y, z) = x^{n-2} \varphi_{n+1}(y, z), \]
and

$$v\left(\frac{1}{A}, B, \frac{D}{A}; \beta\right) = \left(\frac{1}{A}, B, \frac{D}{A}; \beta\right), v_2\left(\frac{1}{A}, B, \frac{D}{A}; \beta\right), \ldots, v_{n-1}\left(\frac{1}{A}, B, \frac{D}{A}; \beta\right)$$

with

$$v_t(x, y, z; \psi_{r+2}) = x'v_t(y; z) \quad \text{for } 1 \leq t \leq n - 3,$$

$$v_{n-2}(x, y, z; \psi_{n+1} + \psi_n(y; z)),$$

$$v_{n-1}(x, y, z) = x'\psi_{n+1}(y, z),$$

respectively.

Here are the main properties of the operators \(\rho\) and \(v\) used in this article:

1. \(\rho(1, 0, \cdot)\) is the identity operator;
2. \(\rho\left(\frac{1}{x}, \frac{B}{x}, \frac{A}{x}; z\right) = \rho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; z\right);\)
3. If \(x' = \rho\left(\frac{1}{x}, \frac{B}{x}, \frac{D}{x}; \beta\right)\), then \(x = \rho(A, -\frac{B}{A}; x').\)

4. CLASSIFICATION PROCEDURE

**Definition 4.1.** An action of algebraic group \(G\) on a variety \(X\) is a morphism \(\sigma : G \times X \rightarrow X\) with:

(i) \(\sigma(e, x) = x\), where \(e\) is an identity element of \(G\) and \(x \in X\);

(ii) \(\sigma(g, \sigma(h, x)) = \sigma(gh, x)\), for all \(g, h \in G\) and \(x \in X\).

One writes \(gx\) for \(\sigma(g, x)\) and call \(X\) a \(G\)-variety. \(O(x) = \{y \in X \mid \exists g \in G, y = gx\}\) is the orbit of \(x\). A function \(f : X \rightarrow K\) is said to be invariant if \(f(gx) = f(x)\) for all \(g \in G\) and \(x \in X\).

We consider the case when \(G = G_{ad}\) and \(X = Lb_{n+1}\). Then the orbits with respect to the action of \(G = G_{ad}\) on \(X = Lb_{n+1}\) consist of all isomorphic to each other algebras.

**4.1. Classification Algorithm and Invariants for \(FLb_{n+1}\)**

Consider the following representation of \(FLb_{n+1} : FLb_{n+1} = U \cup F\), where

\[U = \{L(x) \in FLb_{n+1} : x_3 \neq 0\} \quad \text{and} \quad F = \{L(x) \in FLb_{n+1} : x_3 = 0\}.\]

Then \(U\) can be represented as a disjoint union of the subsets

\[U_1 = \{L(x) \in U : x_4 \neq -2x_3^2\} \quad \text{and} \quad F_1 = \{L(x) \in U : x_4 = -2x_3^2\}.\]
Theorem 4.1.

(i) Two algebras \( L(x) \) and \( L(x') \) from \( U_i \) are isomorphic if and only if

\[
\rho_i \left( \frac{2x_3}{x_4 + 2x_3^2}, \frac{x_4}{2x_3^2} ; x \right) = \rho_i \left( \frac{2x'_3}{x'_4 + 2x'_3^2}, \frac{x'_4}{2x'_3^2} ; x' \right),
\]

whenever \( i = 3, 4, \ldots, n - 1 \).

(ii) For any \( (a_3, a_4, \ldots, a_{n-1}) \in \mathbb{C}^{n-3} \), there is an algebra \( L(x) \) from \( U_i \) such that

\[
\rho_i \left( \frac{2x_3}{x_4 + 2x_3^2}, \frac{x_4}{2x_3^2} ; x \right) = a_i, \quad i = 3, 4, \ldots, n - 1.
\]

Proof. (i). “If” part. Let two algebras \( L(x) \) and \( L(x') \) be isomorphic. Then there exist \( A, B \in \mathbb{C} \) such that \( A(A + B) \neq 0 \) and \( x' = \rho \left( \frac{1}{A}, \frac{1}{B} ; x \right) \). Hence, \( x = \rho(A, -\frac{B}{A+B} ; x') \). Consider the algebra \( L(x^0) \), where \( x^0 = \rho \left( \frac{1}{A_0}, \frac{1}{B_0} ; x \right) \) and \( A_0 = \frac{x^0 + 2x^0_3}{2x^0_3}, \quad B_0 = \frac{2(x^0 + 2x^0_3)}{4x^0_3} \). Then \( x^0 = \rho \left( \frac{2x_3}{x_4 + 2x_3^2}, \frac{x_4}{2x_3^2} ; x \right) = \rho \left( \frac{1}{A_0}, \frac{1}{B_0} ; \rho(A, -\frac{B}{A+B} ; x') \right) = \rho \left( \frac{A_0}{A_0}, \frac{B_0A - A_0B}{A_0(A+B)} ; x' \right) \). It is easy to check that \( A_0 = \frac{2x_3}{x_4 + 2x_3^2} \) and \( B_0A - A_0B = \frac{x_3}{2x_3^2} \).

Therefore,

\[
\rho \left( \frac{2x_3}{x_4 + 2x_3^2}, \frac{x_4}{2x_3^2} ; x \right) = \rho \left( \frac{2x'_3}{x'_4 + 2x'_3^2}, \frac{x'_4}{2x'_3^2} ; x' \right)
\]

and, hence,

\[
\rho_i \left( \frac{2x_3}{x_4 + 2x_3^2}, \frac{x_4}{2x_3^2} ; x \right) = \rho_i \left( \frac{2x'_3}{x'_4 + 2x'_3^2}, \frac{x'_4}{2x'_3^2} ; x' \right),
\]

for all \( i = 3, 4, \ldots, n - 1 \).

This procedure can be shown schematically as follows:

\[
(A, B) \xrightarrow{\alpha} (A_0B_0) \xrightarrow{\alpha^0} \left( A_0A^{-1}, \frac{B_0A - A_0B}{A_0(A+B)} \right).
\]

“Only if” part. Let the equalities

\[
\rho_i \left( \frac{2x_3}{x_4 + 2x_3^2}, \frac{x_4}{2x_3^2} ; x \right) = \rho_i \left( \frac{2x'_3}{x'_4 + 2x'_3^2}, \frac{x'_4}{2x'_3^2} ; x' \right), \quad i = 3, 4, \ldots, n - 1
\]

hold. Then it is easy to see that

\[
\rho_i \left( \frac{2x_3}{x_4 + 2x_3^2}, \frac{x_4}{2x_3^2} ; x \right) = \rho_i \left( \frac{2x'_3}{x'_4 + 2x'_3^2}, \frac{x'_4}{2x'_3^2} ; x' \right) \quad \text{for } i = 1, 2
\]

as well and, therefore, \( \rho \left( \frac{2x_3}{x_4 + 2x_3^2}, \frac{x_4}{2x_3^2} ; x \right) = \rho \left( \frac{2x'_3}{x'_4 + 2x'_3^2}, \frac{x'_4}{2x'_3^2} ; x' \right) \), which means that the algebras \( L(x) \) and \( L(x') \) are isomorphic.
(ii). The system of equations
\[
\rho_i \left( \frac{2x_3}{x_4 + 2x_3}, \frac{x_4}{2x_3}; x \right) = a_i, \quad 3 \leq i \leq n - 1;
\]
where \((a_3, a_4, \ldots, a_{n-1})\) is given and \(x = (x_3, x_4, \ldots, x_{n-1}, \theta)\) is unknown, has a solution as far as for any \(3 \leq i \leq n - 1\) in \(\rho_i \left( \frac{2x_3}{x_4 + 2x_3}, \frac{x_4}{2x_3}; x \right)\) only variables \(x_3, x_4, \ldots, x_i\) occur and each of these equations is a linear equation with respect to the last variable occurred in it. Hence, making each of \(x_i\) the subject of (2), where \(i = 3, \ldots, n - 1\), one can find the required algebra \(L(x)\). \qed

Let us now consider the isomorphism criterion for \(F_1\). This set in its turn can be written as a disjoint union of the subsets
\[
V_1 = \{L(x) \in F_1 : x_5 \neq 5x_3^4\} \quad \text{and} \quad G_1 = \{L(x) \in F_1 : x_5 = 5x_3^4\},
\]
and \(V_1\) can be represented as a disjoint union of the subsets
\[
U_2 = \{L(x) \in V_1 : x_6 + 6x_3x_5 - 16x_3^4 \neq 0\} \quad \text{and} \quad G_2 = \{L(x) \in V_1 : x_6 + 6x_3x_5 - 16x_3^4 = 0\}.
\]

Then the isomorphism criterion for \(U_2\) can be spelled out as follows.

**Theorem 4.2.**

(i) Two algebras \(L(x)\) and \(L(x')\) from \(U_2\) are isomorphic if and only if
\[
\rho_i \left( \frac{5x_3^3 - x_3}{x_6 + 6x_3x_5 - 16x_3^4}, \frac{x_6 + 7x_3x_5 - 21x_3^4}{x_3 (5x_3^3 - x_3)}; x \right) = \rho_i \left( \frac{5x_3^3 - x_3'}{x_6' + 6x_3'x_5' - 16x_3'^4}, \frac{x_6' + 7x_3'x_5' - 21x_3'^4}{x_3' (5x_3'^3 - x_3')}; x' \right)
\]
for \(i = 4, \ldots, n - 1\).

(ii) For any \((a_4, \ldots, a_{n-1}) \in \mathbb{C}^{n-4}\), there is an algebra \(L(x)\) from \(U_2\) such that
\[
\rho_i \left( \frac{5x_3^3 - x_3}{x_6 + 6x_3x_5 - 16x_3^4}, \frac{x_6 + 7x_3x_5 - 21x_3^4}{x_3 (5x_3^3 - x_3)}; x \right) = a_i, \quad i = 4, 5, \ldots, n - 1.
\]

Proof can be carried out with minor changing in the proof of the Theorem 4.1. As for the subsets \(F, G_1,\) and \(G_2\), the isomorphism criteria for them can be treated likewise.

### 4.2. Classification Algorithm and Invariants for \(SLb_{n+1}\)

In this section we consider \(SLb_{n+1}\). The classification algorithm in this case works effectively as well. However, in this case, instead of the representation \(\rho\) we
have to use the representation \( \nu \) (see Section 3). To illustrate it, we represent \( SLb_{n+1} \) as a union of two stable subsets:

\[
SLb_{n+1} = U \cup F,
\]

where

\[
U = \{ L(\beta) : (4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_6 + \beta_4^3)(4\beta_3\beta_5 - 5\beta_5^2) \neq 0 \}
\]

and

\[
F = \{ L(\beta) : (4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_6 + \beta_4^3)(4\beta_3\beta_5 - 5\beta_5^2) = 0 \}.
\]

**Theorem 4.3.**

(i) Two algebras \( L(\beta) \) and \( L(\beta') \) from \( U \) are isomorphic if and only if

\[
\nu_i \left( \frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_6 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_5^2}, \frac{\beta_4}{2\beta_3^2}, \frac{1}{\beta_3}; \beta \right) = \nu_i \left( \frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_6 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_5^2}, \frac{\beta_4}{2\beta_3^2}, \frac{1}{\beta_3}; \beta' \right),
\]

whenever \( i = 3, 4, \ldots, n - 1 \).

(ii) For any \((b_3, b_4, \ldots, b_{n-1}) \in \mathbb{C}^{n-3}\), there is an algebra \( L(\beta) \) from \( U \) such that

\[
\nu_i \left( \frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_6 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_5^2}, \frac{\beta_4}{2\beta_3^2}, \frac{1}{\beta_3}; \beta \right) = b_i, \quad i = 3, 4, \ldots, n - 1.
\]

**Proof.** i). Let \( L(\beta) \) and \( L(\beta') \) be isomorphic. Then there exist \( A, B, D \in \mathbb{C} \) such that \( AD \neq 0 \) and \( \beta' = \nu(\frac{1}{\beta_0}, \frac{b_0}{\beta_0}, 0; \beta) \). Consider the algebra \( L(\beta_0) \), where \( \beta_0 = \nu(\frac{1}{\beta_0}, \frac{b_0}{\beta_0}, 0; \beta) \), and

\[
A_0 = \frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_6 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_5^2}, \quad B_0 = \frac{\beta_4(4\beta_3\beta_5 - 5\beta_5^2)}{2\beta_3^2(4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_6 + \beta_4^3)}, \quad D_0 = \frac{4\beta_3\beta_5 - 5\beta_5^2}{\beta_3(4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_6 + \beta_4^3)}.
\]

Since \( \beta = \nu(A, \frac{B}{D}, A; \beta') \), then

\[
\beta_0 = \nu \left( \frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_6 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_5^2}, \frac{\beta_4}{2\beta_3^2}, \frac{1}{\beta_3}; \beta \right) = \nu \left( \frac{1}{A_0}, \frac{B_0}{A_0}, \frac{D_0}{A_0}, \nu \left( A, \frac{B}{D}, A; \beta' \right) \right) = \nu \left( A, \frac{B}{A_0}, \frac{D_0A}{A_0D}, \beta' \right).
\]
One can easily check that
\[ \frac{A}{A_0} = \frac{4\beta^2_5\beta_6 - 12\beta_3\beta_4\beta_5 + \beta^3_4}{4\beta_3\beta_6 - 5\beta^2_4}, \quad \frac{B_0A - A_0B}{A_0D} = \frac{\beta^4_3}{2\beta^2_3}, \quad \text{and} \quad \frac{D_0A}{A_0D} = \frac{1}{\beta^4_3}. \]

Therefore,
\[ v \left( \frac{4\beta^2_5\beta_6 - 12\beta_3\beta_4\beta_5 + \beta^3_4}{4\beta_3\beta_6 - 5\beta^2_4}, \frac{\beta^4_3}{2\beta^2_3}, \frac{1}{\beta^4_3} ; \beta \right) = v \left( \frac{4\beta^2_5\beta_6 - 12\beta_3\beta_4\beta_5 + \beta^3_4}{4\beta_3\beta_6 - 5\beta^2_4}, \frac{\beta^4_3}{2\beta^2_3}, \frac{1}{\beta^4_3} ; \beta \right) \]
and
\[ v \left( \frac{4\beta^2_5\beta_6 - 12\beta_3\beta_4\beta_5 + \beta^3_4}{4\beta_3\beta_6 - 5\beta^2_4}, \frac{\beta^4_3}{2\beta^2_3}, \frac{1}{\beta^4_3} ; \beta \right) = v \left( \frac{4\beta^2_5\beta_6 - 12\beta_3\beta_4\beta_5 + \beta^3_4}{4\beta_3\beta_6 - 5\beta^2_4}, \frac{\beta^4_3}{2\beta^2_3}, \frac{1}{\beta^4_3} ; \beta \right), \]
for all \( i = 3, 4, \ldots, n - 1. \)

This procedure can be shown schematically by the following picture:

\[ \begin{array}{c}
\beta \\
\left( \frac{A_0}{A}, \frac{B_0}{B}, \frac{D_0}{D} \right) \\
\left( \frac{A_0}{A}, \frac{B_0}{B}, \frac{D_0}{D} \right)
\end{array} \]

Conversely, let the equalities
\[ v_i \left( \frac{4\beta^2_5\beta_6 - 12\beta_3\beta_4\beta_5 + \beta^3_4}{4\beta_3\beta_6 - 5\beta^2_4}, \frac{\beta^4_3}{2\beta^2_3}, \frac{1}{\beta^4_3} ; \beta \right) = v_i \left( \frac{4\beta^2_5\beta_6 - 12\beta_3\beta_4\beta_5 + \beta^3_4}{4\beta_3\beta_6 - 5\beta^2_4}, \frac{\beta^4_3}{2\beta^2_3}, \frac{1}{\beta^4_3} ; \beta \right) \]
hold for \( i = 3, 4, \ldots, n - 1. \) Then it is easy to see that
\[ v_i \left( \frac{4\beta^2_5\beta_6 - 12\beta_3\beta_4\beta_5 + \beta^3_4}{4\beta_3\beta_6 - 5\beta^2_4}, \frac{\beta^4_3}{2\beta^2_3}, \frac{1}{\beta^4_3} ; \beta \right) = v_i \left( \frac{4\beta^2_5\beta_6 - 12\beta_3\beta_4\beta_5 + \beta^3_4}{4\beta_3\beta_6 - 5\beta^2_4}, \frac{\beta^4_3}{2\beta^2_3}, \frac{1}{\beta^4_3} ; \beta \right) \]
for \( i = 1, 2 \) as well and, therefore,
\[ v \left( \frac{4\beta^2_5\beta_6 - 12\beta_3\beta_4\beta_5 + \beta^3_4}{4\beta_3\beta_6 - 5\beta^2_4}, \frac{\beta^4_3}{2\beta^2_3}, \frac{1}{\beta^4_3} ; \beta \right) = v \left( \frac{4\beta^2_5\beta_6 - 12\beta_3\beta_4\beta_5 + \beta^3_4}{4\beta_3\beta_6 - 5\beta^2_4}, \frac{\beta^4_3}{2\beta^2_3}, \frac{1}{\beta^4_3} ; \beta \right) \]
that means the algebras \( L(\beta) \) and \( L(\beta') \) are isomorphic.

Part ii) is similar to that of the Theorem 4.1.

In regard to the set \( F \), it can be split into subsets and the algorithm can be applied with \( v \), instead of \( \rho \), by using the properties of \( v \).

### 4.3. Simplifications in \( \text{TLb}_{n+1} \)

In this section we treat filiform Leibniz algebras whose natural gradation is an algebra from \( \text{NGF}_3 \). This class has been denoted as \( \text{TLb}_{n+1} \). Here we clarify the table
of multiplications of algebras from \( T L b_{n+1} \) and investigate the behavior of structure constants under the base change. We recall that \((n+1)\)-dimensional filiform Lie algebras are in \( T L b_{n+1} \). A study of \( T L b_{n+1} \) was initiated in Omirov et al. [17].

**Proposition 4.1.**

\[
TL b_{n+1} = \begin{bmatrix}
[e_i, e_0] &= e_{i+1}, & 1 \leq i \leq n-1, \\
[e_0, e_i] &= -e_{i+1}, & 2 \leq i \leq n-1, \\
[e_0, e_0] &= b_{0,0}e_n, \\
[e_0, e_1] &= -e_2 + b_{0,1}e_n, \\
[e_1, e_1] &= b_{1,1}e_n, \\
[e_1, e_i] &= a_{1,i}e_{i+1} + \cdots + a_{n-i+1-i}e_{n-1} + b_{1,i}e_n, & 1 \leq i < j \leq n-2, \\
[e_1, e_j] &= -[e_j, e_i], & 1 \leq i < j \leq n-1, \\
[e_i, e_{n-i}] &= -[e_{n-i}, e_i] = (-1)^ib_{i,n-i}e_n, & 1 \leq i \leq n-1, \\
\end{bmatrix}
\]

(5)

where \( a_{i,j}, b_{i,j} \in \mathbb{C} \) and \( b_{i,n-i} = b \), whenever \( 1 \leq i \leq n-1, b \in \{0,1\} \) for odd \( n \) and \( b = 0 \) for even \( n \).

**Proof.** Let \( L \in TL b_{n+1} \) and \( e_0, e_1, \ldots, e_n \) be a basis of \( L \). Then due to Theorem 2.2 \([e_i, e_j] \in \text{span}\{e_{i+j+1}, \ldots, e_n\}\) for any \( i, j \neq 0 \).

Then

\[
[e_i, e_0] = e_{i+1} + (\ast)e_{i+2} + \cdots + (\ast)e_n, \quad 1 \leq i \leq n-1.
\]

Putting \( e'_i = e_i, e'_0 = e_0, e'_{i+1} := [e'_i, e'_0] \), we may assume that \([e_i, e_0] = e_{i+1}, 1 \leq i \leq n-1\).

Now consider

\[
[e_0, e_i] = -e_{i+1} + x_{0,i+2}e_{i+2} + x_{0,i+3}e_{i+3} + \cdots + x_{0,i}e_n, \quad 1 \leq i \leq n-1.
\]

Then we get

\[
[e_i, e_0] + [e_0, e_i] = x_{0,i+2}e_{i+2} + x_{0,i+3}e_{i+3} + \cdots + x_{0,i}e_n, \quad 1 \leq i \leq n-1.
\]

(6)

Note that the Leibniz identity implies that \([x, y] + [y, x] \in \mathfrak{sp}(L)\), for any \( x, y \in L \), where \( \mathfrak{sp}(L) \) is the right annihilator of \( L \). Therefore, if we multiply the both sides of (6) from the right-hand side \((n-i-2)\) times by \( e_0 \), we obtain \( x_{0,i}^{n-i} = 0 \). Substituting and repeating it, we get

\[
x_{0,i}^{n-i+2} = 0, \quad 2 \leq i \leq n-1.
\]

Applying the above to \([e_i, e_j], 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\), we get \([e_i, e_i] = x_{i,i}e_n\).

The chain of equalities

\[
[e_0, e_i] = [e_0, [e_{i-1}, e_0]] = [[e_0, e_{i-1}], e_0] - [[e_0, e_0], e_{i-1}]
= [-e_i + x_{0,i-1}e_n, e_0] = -[e_i, e_0] = -e_{i+1}
\]
leads to \([e_i, e_0] = -[e_0, e_i] = e_{i+1}\) for \(2 \leq i \leq n - 1\), i.e., \([e_0, x] = -[x, e_0]\) for any \(x \in L^2\).

We claim that

\[ [e_i, e_j] = -[e_j, e_i], \quad 1 \leq i < j \leq n - 1. \quad (7) \]

The induction by \(i\) for any \(j\) and the following chain of equalities:

\[
[e_i, e_{j+1}] = [e_i, [e_j, e_0]] = [[e_i, e_j], e_0] - [[e_i, e_0], e_j] = (\text{since } [e_i, e_j] \in L^2) \\
= -[e_0, [e_i, e_j]] + [[e_0, e_j] - \xi_0 e_n, e_j] = -[e_0, [e_i, e_j]] + [[e_0, e_j], e_j] \\
= -[[e_0, e_i], e_j] + [[e_0, e_j], e_j] + [[e_0, e_i], e_j] = -[e_{j+1}, e_i], \quad 1 \leq j \leq n - 1
\]

show (7).

The above observations lead to the required table of multiplication of \(L \in TL_{n+1}\).

Let \(L \in TL_{n+1}\). The subspace spanned by \(\{e_n\}\) is an ideal of \(L\) and the quotient algebra \(L/\langle e_n \rangle\) is the \(n\)-dimensional filiform Lie algebra \(\mu_n\) with the composition law

\[
[e_i, e_0] = e_{i+1}, \\
[e_i, e_j] = a^1_{i, j} e_{i+j+1} + \cdots + a^n_{i, j} e_{n-1}, \quad 1 \leq i < j \leq n - 1.
\]

Moreover, \(e_n\) is in the center of \(L\). Therefore, \(L\) can be considered as a Leibniz central extension of \(\mu_n\).

**Lemma 4.1.** Let \(L \in TL_{n+1}\). Then

\[
\sum_{s=1}^{n-(i+j+k+1)} a^s_{i, j, k} b_{i+j+k+s} = \sum_{s=1}^{n-(i+j+k+1)} (a^s_{i, j, k} b_{i+j+k+s} - a^s_{i, j} b_{i+j+s+k}). \quad (8)
\]

**Proof.** The Leibniz identity for \(e_i, e_j,\) and \(e_k\) gives the required relations between the structure constants. Indeed,

\[
[e_i, [e_j, e_k]] = \left[ e_i, \sum_{s=1}^{n-(i+j+k+1)} a^s_{i, j, k} e_{i+j+k+s} + b_{i, j, k} e_n \right] \\
= \sum_{s=1}^{n-(i+j+k+1)} a^s_{i, j, k} \left( \sum_{t=1}^{n-(i+j+k+s+1)} a^t_{i, j+k+s} e_{i+j+k+s+t} + b_{i, j+k+s} e_n \right),
\]

\[
[[e_i, e_j], e_k] = \left[ \sum_{s=1}^{n-(i+j+1)} a^s_{i, j} e_{i+j+s} + b_{i, j} e_n, e_k \right] \\
= \sum_{s=1}^{n-(i+j+k+1)} a^s_{i, j} \left( \sum_{t=1}^{n-(i+j+s+k+1)} a^t_{i+j+s+k} e_{i+j+s+k+t} + b_{i+j+s+k} e_n \right).
\]
\[ [e_i, e_j] = \sum_{s=1}^{n-i+j+1} a^i_s b^i_{i+k+s} e_{i+k+s} + b^i_{i+k+s} e_n, e_j \]

and then it implies that

\[ \sum_{s=1}^{n-(i+j+k+1)} a^i_s b^i_{i+k+s} = \sum_{s=1}^{n-(i+j+k+1)} (a^i_s b^i_{i+j+k, s} - a^i_s b^i_{i+k+s}). \]

Here are several useful remarks regarding (8) that permit much simplify the multiplication table of \( TLb_{n+1} \):

1. It is symmetric with respect to \( i, j, k \) (since \( a^i_{s,t} = -a^i_{t,s} \) and \( b^i_{s,t} = -b^i_{t,s} \) for any \( s \) and \( t \), except for \( (s, t) = (0, 0), (1, 1), (0, 1), (1, 0) \)).

2. In the case when \( (i, j, k) = (0, j, k) \), we get

\[ \sum_{s=1}^{n-(j+k+1)} a^j_s b^j_{0,i+k+s} = \sum_{s=1}^{n-(j+k+1)} (a^j_s b^j_{0,i+k,s} - a^j_s b^j_{0,k+s}). \]

where \( j \neq 0, k \neq 0 \).

3. Since \( a^0_{0,s} = 0 \) as \( s \neq 1 \) and \( a^0_{0,i} = 1 \), we get

\[ a^1_{j,k} b^1_{0,i+k+1} + a^1_{j,k} b^1_{0,i+k+2} + \cdots + a^{n-(j+k+1)}_{j,k} b^1_{0,n-1} = -b^j_{i+1,k} + b^j_{i+1,j}. \]

4. Since \( b^0_{0,s} = 0 \) as \( t = 2, \ldots, n-2 \) and \( b^0_{n-1} = 1 \), it implies that

\[ a^{n-(j+k+1)}_{j,k} = b^j_{i+k+1,k} - b^j_{i+k+1,j}. \]

for \( k = j + 1, j + 2, \ldots, n-j-2 \) and \( j = 1, 2, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor \).

**Lemma 4.2.** Let \( L \in TLb_{n+1} \). Then

\[ [e_i, e_{j+k}] = \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} [e_{i+k-s}, e_{j}] R_{e_0}^{s}, \]

where \( 1 \leq i, j, k \leq n \) and \( yR_s = [y, x] \) is the right multiplication operator on \( L \).

**Proof.** The proof will be proceed by the induction with respect to \( k \). Let \( k = 1 \). Then \( [e_i, e_{j+1}] = [e_i, [e_j, e_0]] = -[e_{j+1}, e_j] + [[e_i, e_j], e_0], \) i.e., (9) holds at \( k = 1 \). This is a base of the induction. Then the following chain of equalities lead to the claim:

\[ [e_i, e_{j+k+1}] = \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} [e_{i+k-s}, e_j] R_{e_0}^{s+1} - \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} [e_{i+k+1-s}, e_j] R_{e_0}^{s}. \]
\[\begin{align*}
&= -\sum_{i=1}^{k+1} (-1)^{k-s} \binom{k}{s-1} \left[ e_{i+k+1-s}, e_j \right] R_0^e_i - \sum_{i=0}^{k} (-1)^{k-s} \binom{k}{s} \left[ e_{i+k+1-s}, e_j \right] R_0^e_i \\
&= \sum_{i=1}^{k} (-1)^{k+1-s} \binom{k}{s-1} \left( \binom{k}{s} + \binom{k-1}{s-1} \right) \left[ e_{i+k+1-s}, e_j \right] R_0^e_i \\
&\quad + \left[ e_{i+k+1}, e_j \right] R_0^e_{k+1} - (-1)^k \left[ e_{i+k+1}, e_k \right] \\
&= \sum_{i=0}^{k+1} (-1)^{k+1-s} \binom{k+1}{s} \left[ e_{i+k+1-s}, e_j \right] R_0^e_i. 
\end{align*}\]

The following proposition specifies elements of \(G_{ad}\) corresponding to the structure of \(L \in TL_{n+1}\).

**Proposition 4.2.** Let \(L \in TL_{n+1}\) and \(f\) be an adapted transformation of \(L\). Then \(f\) can be represented as follows:

\[
\begin{align*}
\sigma(e_0) &= e_0' = \sum_{i=0}^{n} A_i e_i, \\
\sigma(e_1) &= e_1' = \sum_{i=1}^{n} B_i e_i, \\
\sigma(e_i) &= e_i' = [\sigma(e_{i-1}), \sigma(e_0)], \quad 2 \leq i \leq n,
\end{align*}
\]

\(A_0, A_j, B_j, (i, j = 1, \ldots, n)\) are complex numbers, and \(A_0 B_1 (A_0 + A_1 b) \neq 0\).

**Proof.** Since a filiform Leibniz algebra is 2-generated, it is sufficient to consider the adapted action of \(f\) on the generators \(e_0, e_1\):

\[
\begin{align*}
\sigma(e_0) &= e_0' = \sum_{i=0}^{n} A_i e_i, \quad \text{and} \quad \sigma(e_1) = e_1' = \sum_{i=0}^{n} B_i e_i.
\end{align*}
\]

Then \(\sigma(e_i) = [\sigma(e_{i-1}), \sigma(e_0)] = A_0^{-2} (A_1 B_0 - A_0 B_1) e_i + \sum_{j=3}^{k} (\ast) e_j, \quad 2 \leq i \leq n\). Note that \(A_0 \neq 0, (A_1 B_0 - A_0 B_1) \neq 0\); otherwise, \(\sigma(e_0) = 0\). The condition \(A_0 B_1 (A_0 + A_1 b) \neq 0\) appears naturally since \(f\) is not singular.

Let now consider \([\sigma(e_1), \sigma(e_2)] = B_0 (A_1 B_0 - A_0 B_1) e_3 + \sum_{j=4}^{n} (\ast) e_j\). Then for the basis \([\sigma(e_0), \sigma(e_1), \ldots, \sigma(e_n)]\) to be adapted, \(B_0 (A_1 B_0 - A_0 B_1)\) must not be zero. But according to the above observation, \((A_1 B_0 - A_0 B_1) \neq 0\). Therefore, \(B_0 = 0\). \(\square\)

The following elements of \(G_{ad}\) are said to be elementary with respect to the structure of \(L \in TL_{n+1}\):

\[
\begin{align*}
\sigma(b, k) &= \begin{cases} 
\sigma(e_0) = e_0, \\
\sigma(e_i) = e_i + b e_k, \quad &b \in \mathbb{C}, \quad 2 \leq k \leq n, \\
\sigma(e_{i+1}) = [\sigma(e_i), \sigma(e_0)], \quad &1 \leq i \leq n - 1.
\end{cases}
\end{align*}
\]
Proposition 4.4. The transformations \( \tau(a, k) = \begin{cases} f(e_0) = e_0 + a e_k, & a \in \mathbb{C}, \ 1 \leq k \leq n, \\ f(e_i) = e_i, & 1 \leq i \leq n - 1, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \leq i \leq n - 1. \end{cases} \)

Lemma 4.3. Let \( v(a, b) = \begin{cases} f(e_0) = a e_0, & a, b \in \mathbb{C}^*, \\ f(e_1) = b e_1, & a, b \in \mathbb{C}^*, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \leq i \leq n - 1. \end{cases} \)

Proposition 4.3. Let \( f \) be an adapted transformation of \( L \). Then it can be represented as a composition:

\[
\begin{align*}
f &= \tau(A_n, n) \circ \tau(A_{n-1}, n-1) \circ \cdots \circ \tau(A_2, 2) \circ \sigma(B_n, n) \circ \sigma(B_{n-1}, n-1) \\
&\quad \circ \cdots \circ \sigma(B_2, 2) \circ \tau(A_1, 1) \circ v(A_0, B_1).
\end{align*}
\]

Proof. The proof is straightforward. \( \square \)

Proposition 4.4. The transformations

\[
g = \tau(A_n, n) \circ \tau(A_{n-1}, n-1) \circ \tau(A_{n-2}, n-2) \circ \tau(A_{n-3}, n-3) \circ \tau(A_{n-4}, n-4) \\
\circ \sigma(B_n, n) \circ \sigma(B_{n-1}, n-1) \circ \sigma(B_{n-2}, n-2) \circ \sigma(B_{n-3}, n-3), \quad \text{if } n \text{ even,}
\]

and

\[
g = \tau(A_n, n) \circ \tau(A_{n-1}, n-1) \circ \tau(A_{n-2}, n-2) \circ \tau(A_{n-3}, n-3) \\
\circ \sigma(B_n, n) \circ \sigma(B_{n-1}, n-1) \circ \sigma(B_{n-2}, n-2), \quad \text{for odd } n
\]
do not change the structure constants of algebras from \( TLb_{n+1} \).

Remark. From the propositions above, one easily can see that the class \( TLb_{n+1} \) is less yieldable to simplification of adapted transformations than the first two classes. Nevertheless, the following lemma tracks out the behavior of some structure constants.

Lemma 4.3. Let \( L \) and \( L' \) be filiform Leibniz algebras from \( TLb_{n+1} \) with the parameters \((b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, \ldots, b_{i,j})\) and \((b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2}, \ldots, b'_{i,j})\), respectively, where \( 1 \leq j \leq n - 1 \). Suppose that \( L' \) is obtained from \( L \) by the adapted base change. Then

\[
\begin{align*}
b'_{0,0} &= \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^2 B_1 (A_0 + A_1 b)} \\
b'_{0,1} &= \frac{A_0 b_{0,1} + 2 A_1 b_{1,1}}{A_0^2 (A_0 + A_1 b)}, \\
b'_{1,1} &= \frac{B_1 b_{1,1}}{A_0^2 (A_0 + A_1 b)}.
\end{align*}
\]
Proof. Consider the product \([f(e_0), f(e_0)] = b'_0,0 f(e_n)\). Equating the coefficients of \(e_n\) in it, we get

\[ A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1} = b'_0,0 A_0^{n-2} B_1 (A_0 + A_1 b). \]

Then \(b'_0,0 = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^{n-2} B_1 (A_0 + A_1 b)}\).

The product \([f(e_1), f(e_1)] = b'_{1,1}, f(e_n)\) yields

\[ b'_{1,1} = \frac{B_1 b_{1,1}}{A_0^{n-2} (A_0 + A_1 b)}. \]

Consider the equality

\[ b'_0,1 f(e_n) = [f(e_1), f(e_0)] + [f(e_0), f(e_1)]. \]

Then \(b'_0,1 A_0^{n-2} B_1 (A_0 + A_1 b) = A_0 B_1 b_{0,1} + 2 A_1 B_1 b_{1,1}\), and it implies that

\[ b'_0,1 = \frac{A_0 b_{0,1} + 2 A_1 b_{1,1}}{A_0^{n-2} (A_0 + A_1 b)}. \]

The complete implementation of the procedure for some low dimensional cases will be given in the next section.

5. APPLICATIONS

The objective of this section is to provide the isomorphism classes of complex filiform Leibniz algebras \(Lb_n\) for \(n = 5, 6\).

For the computational purpose, we establish the following notations and conventions:

- \(\Delta_4 = x_4 + 2 x_3^2\), \(\Delta_5 = x_5 - 5 x_3^3\), \(\Theta_i = \theta - x_i\), \(i = 4, 5\), and the letters \(\Delta_4\), \(\Delta_5\), \(\Theta_4\), and \(\Theta_5\) with \(\prime\) (prime) denote the same expression depending on parameters \(x'_4, x'_5, \theta'\). Notice that \(\Delta_4 = x_i\) \((i = 4, 5)\) as \(x_3 = 0\).
- \(\Lambda = 4 \beta_4 \beta_5 - 5 \beta_4^2\) and \(\Lambda' = 4 \beta'_4 \beta'_5 - 5 \beta'_4^2\).
- \(\Delta = 4 b_{0,0} b_{1,1} - b_{0,1}^2\) and \(\Delta' = 4 b'_{0,0} b'_{1,1} - b'_{0,1}^2\).

5.1. The Isomorphism Classes in \(FLb_5\) and \(FLb_6\)

5.1.1. Dimension 5. The class \(FLb_5\) can be represented as a disjoint union of the following subsets:

\[ FLb_5 = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6 \cup U_7, \]

where

\[ U_1 = \{ L(x) \in FLb_5 : x_3 \neq 0, \Delta_4 \neq 0 \}, \]

\[ U_2 = \{ L(x) \in FLb_5 : x_3 \neq 0, \Delta_4 = 0, \Theta_4 \neq 0 \}, \]
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\[ U_3 = \{ L(x) \in FLb_3 : \alpha_3 \neq 0, \Delta_4 = 0, \Theta_4 = 0 \}, \]
\[ U_4 = \{ L(x) \in FLb_3 : \alpha_3 = 0, \Delta_4 \neq 0, \Theta_4 \neq 0 \}, \]
\[ U_5 = \{ L(x) \in FLb_3 : \alpha_3 = 0, \Delta_4 \neq 0, \Theta_4 = 0 \}, \]
\[ U_6 = \{ L(x) \in FLb_3 : \alpha_3 = 0, \Delta_4 = 0, \Theta_4 \neq 0 \}, \]
\[ U_7 = \{ L(x) \in FLb_3 : \alpha_3 = 0, \Delta_4 = 0, \Theta_4 = 0 \}. \]

Now we consider the isomorphism problem for each of these sets separately.

**Proposition 5.1.** Two algebras \( L(l/\alpha r) \) and \( L(l/\alpha r') \) from \( U_1 \) are isomorphic if and only if

\[
\left( \frac{\alpha_3}{\Delta_4} \right)^2 \Theta_4 = \left( \frac{\alpha_3'}{\Delta_4} \right)^2 \Theta_4'
\]

The expression

\[
\left( \frac{\alpha_3}{\Delta_4} \right)^2 \Theta_4
\]

can be taken as a parameter \( \lambda \), and orbits from the set \( U_1 \) can be parameterized as \( L(1, 0, \lambda), \lambda \in \mathbb{C} \).

**Proposition 5.2.** The subsets \( U_2, U_3, U_4, U_5, U_6, \) and \( U_7 \) are single orbits with the representatives \( L(1, -2, 0), L(1, -2, -2), L(0, 1, 0), L(0, 1, 1), L(0, 0, 1) \), and \( L(0, 0, 0) \).

We summarize these all in the following theorem.

**Theorem 5.1.** Let \( L \) be a non-Lie complex filiform Leibniz algebra in \( FLb_5 \). Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras:

1) \( L(0, 0, 0) \):

\[ L^\mathcal{L}_5 = \{ e_0 e_0 = e_2, e_i e_0 = e_0 e_{i+1}, 1 \leq i \leq 3 \}. \]

2) \( L(0, 0, 1) \):

\[ L^\mathcal{L}_5, \quad e_0 e_1 = e_4. \]

3) \( L(0, 1, 1) \):

\[ L^\mathcal{L}_5, \quad e_0 e_1 = e_4, \quad e_1 e_1 = e_4. \]

4) \( L(0, 1, 0) \):

\[ L^\mathcal{L}_5, \quad e_1 e_1 = e_4. \]
5) \(L(1, -2, -2):
\[
L'_5, \quad e_0e_1 = e_3 - 2e_4, \quad e_1e_1 = e_3 - 2e_4, \quad e_2e_1 = e_4.
\]

6) \(L(1, -2, 0):
\[
L'_5, \quad e_0e_1 = e_3, \quad e_1e_1 = e_3 - 2e_4, \quad e_2e_1 = e_4.
\]

7) \(L(1, 0, \lambda):
\[
L'_5, \quad e_0e_1 = e_3 + \lambda e_4, \quad e_1e_1 = e_3, \quad e_2e_1 = e_4, \quad \lambda \in \mathbb{C}.
\]

5.1.2. Dimension 6. The set \(FLb_6\) can be represented as a disjoint union of its subsets as follows:
\[
FLb_6 = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6 \cup U_7 \cup U_8 \cup U_9 \cup U_{10} \cup U_{11},
\]
where
\[
\begin{align*}
U_1 &= \{L(x) \in FLb_6 : x_3 \neq 0, \Delta_4 \neq 0\}, \\
U_2 &= \{L(x) \in FLb_6 : x_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_3 \neq 0\}, \\
U_3 &= \{L(x) \in FLb_6 : x_3 = 0, \Delta_4 \neq 0, \Delta_5 \neq 0\}, \\
U_4 &= \{L(x) \in FLb_6 : x_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_3 = 0\}, \\
U_5 &= \{L(x) \in FLb_6 : x_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_3 \neq 0\}, \\
U_6 &= \{L(x) \in FLb_6 : x_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Theta_3 \neq 0\}, \\
U_7 &= \{L(x) \in FLb_6 : x_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Theta_3 = 0\}, \\
U_8 &= \{L(x) \in FLb_6 : x_3 = 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_3 \neq 0\}, \\
U_9 &= \{L(x) \in FLb_6 : x_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_3 = 0\}, \\
U_{10} &= \{L(x) \in FLb_6 : x_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_3 \neq 0\}, \\
U_{11} &= \{L(x) \in FLb_6 : x_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_3 = 0\}.
\end{align*}
\]

Proposition 5.3. Two algebras \(L(x)\) and \(L(x')\) from \(U_1\) are isomorphic if and only if
\[
\frac{x_3(\Delta_3 + 5x_3\Delta_4)}{\Delta_4^2} = \frac{x'_3(\Delta'_3 + 5x'_3\Delta'_4)}{\Delta'_4^2}
\]
and
\[
\frac{x_3^3\Theta_3}{\Delta_3^3} = \frac{x'_3^3\Theta'_3}{\Delta'_3^3}.
\]
The following two expressions can be taken as parameters $\lambda_1, \lambda_2$:

$$\frac{x_3(\Delta_2 + 5x_3\Delta_4)}{\Delta_4^2}, \quad \frac{x_3^2\Theta_3}{\Delta_3^3},$$

and orbits in $U_1$ are parameterized as

$$L(1, 0, \lambda_1, \lambda_2), \quad \lambda_1, \lambda_2 \in \mathbb{C}.$$

**Proposition 5.4.** Two algebras $L(x)$ and $L(x')$ from $U_2$ are isomorphic if and only if

$$\frac{\Delta_3^3}{x_3^3\Theta_3^2} = \frac{\Delta_3^3'}{x_3^3\Theta_3'^2}.$$

In the set $U_2$, the expression

$$\frac{\Delta_3^3}{x_3^3\Theta_3^2}$$

can be taken as a parameter and orbits from $U_2$ are parameterized as

$$L(1, -2, \lambda, 2\lambda - 5), \quad \lambda \in \mathbb{C}.$$

**Proposition 5.5.** Two algebras $L(x)$ and $L(x')$ from $U_3$ are isomorphic if and only if

$$\frac{x_3^3\Theta_3}{x_5^3} = \frac{x_3^3\Theta_3'}{x_5^3},$$

and the expression

$$\frac{x_3^3\Theta_3}{x_5^3}$$

can be taken as a parameter and orbits from the set $U_3$ can be represented as a union of orbits with representatives

$$L(0, 1, 1, \lambda), \quad \lambda \in \mathbb{C}.$$

**Proposition 5.6.** Subsets $U_4$, $U_5$, $U_6$, $U_7$, $U_8$, $U_9$, $U_{10}$, and $U_{11}$ are single orbits with the representatives $L(1, -2, 0, 0)$, $L(1, -2, 5, 0)$, $L(0, 1, 0, 1)$, $L(0, 1, 0, 0)$, $L(0, 0, 1, 0)$, $L(0, 0, 1, 1)$, $L(0, 0, 0, 1)$, and $L(0, 0, 0, 0)$, respectively.

**Theorem 5.2.** Let $L$ be a non-Lie complex filiform Leibniz algebra in $FL_{ib6}$. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras:

1) $L(0, 0, 0, 0)$:

$$L_0' = \{e_0e_0 = e_2, e_1e_0 = e_{i+1}, 1 \leq i \leq 4\}.$$
2) \( L(0, 0, 0, 1): \)
\[
L^f_6, \quad e_0 e_1 = e_5.
\]

3) \( L(0, 0, 1, 1): \)
\[
L^f_6, \quad e_0 e_1 = e_5, \quad e_1 e_1 = e_5.
\]

4) \( L(0, 0, 1, 0): \)
\[
L^f_6, \quad e_1 e_1 = e_5.
\]

5) \( L(0, 1, 0, 0): \)
\[
L^f_6, \quad e_0 e_1 = e_4, \quad e_1 e_1 = e_4, \quad e_2 e_1 = e_5.
\]

6) \( L(0, 1, 0, 1): \)
\[
L^f_6, \quad e_0 e_1 = e_4 + e_5, \quad e_1 e_1 = e_4, \quad e_2 e_1 = e_5.
\]

7) \( L(1, -2, 5, 0): \)
\[
L^f_6, \quad e_0 e_1 = e_3 - 2e_4, \quad e_1 e_1 = e_3 - 2e_4 + 5e_5, \quad e_2 e_1 = e_4 - 2e_5, \quad e_3 e_1 = e_5.
\]

8) \( L(1, -2, 0, 0): \)
\[
L^f_6, \quad e_0 e_1 = e_3 - 2e_4, \quad e_1 e_1 = e_3 - 2e_4, \quad e_2 e_1 = e_4 - 2e_5, \quad e_3 e_1 = e_5.
\]

9) \( L(0, 1, 1, \lambda): \)
\[
L^f_6, \quad e_0 e_1 = e_4 + \lambda e_5, \quad e_1 e_1 = e_4 + e_5, \quad e_2 e_1 = e_5, \quad \lambda \in \mathbb{C}.
\]

10) \( L(1, -2, \lambda, 2\lambda - 5): \)
\[
L^f_6, \quad e_0 e_1 = e_3 - 2e_4 + (2\lambda - 5)e_4, \quad e_1 e_1 = e_3 - 2e_4 + \lambda e_5, \quad e_2 e_1 = e_4 - 2e_5,
\]
\[
e_3 e_1 = e_5, \quad \lambda \in \mathbb{C}.
\]

11) \( L(1, 0, \lambda_1, \lambda_2): \)
\[
L^f_6, \quad e_0 e_1 = e_3 + \lambda_1 e_5, \quad e_1 e_1 = e_3 + \lambda_2 e_5, \quad e_2 e_1 = e_4, \quad e_3 e_1 = e_5, \quad \lambda_1, \lambda_2 \in \mathbb{C}.
\]

The Propositions 5.1, 5.3–5.5 are variation of the Theorem 4.1. Propositions 5.2 and 5.6 can be proven by precise indication of base change leading to the representatives.
5.2. The Isomorphism Classes in $SL_b^5$ and $SL_b^6$

5.2.1. Dimension 5. It is easy to see that there is the following representation of $SL_b^5$ as a disjoint union of its subsets as follows:

$$SL_b^5 = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5,$$

where

$U_1 = \{L(\beta) \in SL_b^5 : \beta_3 \neq 0, \gamma - 2\beta_3^2 \neq 0\},$

$U_2 = \{L(\beta) \in SL_b^5 : \beta_3 \neq 0, \gamma - 2\beta_3^2 = 0, \beta_4 \neq 0\},$

$U_3 = \{L(\beta) \in SL_b^5 : \beta_3 = 0, \gamma \neq 0\},$

$U_4 = \{L(\beta) \in SL_b^5 : \beta_3 = 0, \gamma = 0, \beta_4 = 0\},$

$U_5 = \{L(\beta) \in SL_b^5 : \beta_3 = 0, \gamma = 0, \beta_4 \neq 0\}.$

**Proposition 5.7.** Two algebras $L(\beta)$ and $L(\beta')$ from $U_1$ are isomorphic if and only if

$$\gamma/\beta_3^2 = \gamma'/\beta_3'^2.$$

Then orbits from the set $U_1$ can be parameterized as $L(1, 0, \lambda) \; \lambda \in \mathbb{C}$.

**Proposition 5.8.** Subsets $U_2, U_3, U_4,$ and $U_5$ are single orbits with the representatives $L(1, 1, 2), L(0, 0, 1), L(0, 1, 0),$ and $L(0, 0, 0)$, respectively.

**Theorem 5.3.** Let $L$ be a non-Lie complex filiform Leibniz algebra in $SL_b^5$. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras:

1) $L(0, 0, 0)$:

$$L_5^* = \{e_0e_0 = e_2, e_ie_0 = e_{i+1}, 2 \leq i \leq 3\}.$$

2) $L(0, 1, 0)$:

$$L_5^*, \; e_0e_1 = e_4.$$

3) $L(0, 0, 1)$:

$$L_5^*, \; e_1e_1 = e_4.$$

4) $L(1, 1, 2)$:

$$L_5^*, \; e_0e_1 = e_3 + e_4, \; e_1e_1 = 2e_4, \; e_2e_1 = e_4.$$

5) $L(1, 0, \lambda)$:

$$L_5^*, \; e_0e_1 = e_3, \; e_1e_1 = \lambda e_4, \; e_2e_1 = e_4, \; \lambda \in \mathbb{C}.$$
5.2.2. Dimension 6. The set $SLb_6$ can be represented as a disjoint union of its subsets

$$SLb_6 = U_1 \cup U_2 \cup U_3 \cup U_4 \cup U_5 \cup U_6 \cup U_7 \cup U_8 \cup U_9,$$

where

$$U_1 = \{ L(\beta) \in SLb_6 : \beta_3 \neq 0, \gamma \neq 0 \},$$
$$U_2 = \{ L(\beta) \in SLb_6 : \beta_3 \neq 0, \gamma = 0, \Lambda \neq 0 \},$$
$$U_3 = \{ L(\beta) \in SLb_6 : \beta_3 = 0, \gamma = 0, \Lambda = 0 \},$$
$$U_4 = \{ L(\beta) \in SLb_6 : \beta_3 = 0, \beta_4 \neq 0, \gamma \neq 0 \},$$
$$U_5 = \{ L(\beta) \in SLb_6 : \beta_3 = 0, \beta_4 \neq 0, \gamma = 0, \beta_5 \neq 0 \},$$
$$U_6 = \{ L(\beta) \in SLb_6 : \beta_3 = 0, \beta_4 \neq 0, \gamma = 0, \beta_5 = 0 \},$$
$$U_7 = \{ L(\beta) \in SLb_6 : \beta_3 = 0, \beta_4 = 0, \gamma \neq 0 \},$$
$$U_8 = \{ L(\beta) \in SLb_6 : \beta_3 = 0, \beta_4 = 0, \gamma = 0, \beta_5 = 0 \}.$$

**Proposition 5.9.** Two algebras $L(\beta)$ and $L(\beta')$ from $U_1$ are isomorphic if and only if

$$\frac{2\beta_3\beta_4\gamma + \beta_3^2\Lambda}{\gamma^2} = \frac{2\beta'_3\beta'_4\gamma' + \beta'_3^2\Lambda'}{\gamma'^2}.$$

Orbits in $U_1$ can be parameterized as $L(1, 0, \lambda, 1), \lambda \in \mathbb{C}$.

**Proposition 5.10.** Algebras $L(1, 0, 1, 0), L(1, 0, 0, 0), L(0, 1, 0, 1), L(0, 1, 1, 0), L(0, 1, 0, 0), L(0, 0, 1, 0), L(0, 0, 1, 0)$, and $L(0, 0, 0, 0)$ are representatives of the single orbits $U_2, U_3, U_4, U_5, U_6$, and $U_7$, respectively.

**Theorem 5.4.** Let $L$ be a non-Lie complex filiform Leibniz algebra in $SLb_6$. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras:

1) $L(0, 0, 0, 0)$:

$$L'_6 = \{ e_0e_0 = e_2, e_ie_0 = e_{i+1}, 2 \leq i \leq 4 \}.$$

2) $L(0, 0, 1, 0)$:

$$L'_6, \quad e_0e_1 = e_5.$$

3) $L(0, 0, 0, 1)$:

$$L'_6, \quad e_1e_1 = e_5.$$
4) $L(0, 1, 0, 0)$:

\[ L^r_6, \ e_0e_1 = e_4, \ e_2e_1 = e_5. \]

5) $L(0, 1, 1, 0)$:

\[ L^r_6, \ e_0e_1 = e_4 + e_5, \ e_2e_1 = e_5. \]

6) $L(0, 1, 0, 1)$:

\[ L^r_6, \ e_0e_1 = e_4, \ e_1e_1 = e_5, \ e_2e_1 = e_5. \]

7) $L(1, 0, 0, 0)$:

\[ L^r_6, \ e_0e_1 = e_3, \ e_2e_1 = e_4, \ e_3e_1 = e_5. \]

8) $L(1, 0, 1, 0)$:

\[ L^r_6, \ e_0e_1 = e_3 + e_5, \ e_2e_1 = e_4, \ e_3e_1 = e_5. \]

9) $L(1, 0, \lambda, 1)$:

\[ L^r_6, \ e_0e_1 = e_3 + \lambda e_5, \ e_1e_1 = e_5, \ e_2e_1 = e_4, \ e_3e_1 = e_5, \ \lambda \in \mathbb{C}. \]

The Propositions 5.7, 5.9 are variation of the Theorem 4.3. Propositions 5.8 and 5.10 can be proven by precise indication of base change leading to the representatives.

5.3. The Description of $TLb_n$, $n = 5, 6$

5.3.1. 5-Dimensional Case. By virtue of Proposition 4.1, we represent $TLb_5$ as follows:

\[
TLb_5 = \begin{pmatrix}
[e_i, e_0] = e_{i+1}, & 1 \leq i \leq 3, \\
[e_0, e_i] = -e_{i+1}, & 2 \leq i \leq 3, \\
[e_0, e_0] = b_{0,0}e_4, \\
[e_0, e_1] = -e_2 + b_{0,1}e_4, \\
[e_1, e_1] = b_{1,1}e_4, \\
[e_1, e_2] = -[e_2, e_1] = b_{1,2}e_4, \\
b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2} & \in \mathbb{C}.
\end{pmatrix}
\]

Here elements of $TLb_5$ will be denoted by $L(x) = (b_{0,1}, b_{0,1}, b_{1,1}, b_{1,2})$. 


Theorem 5.5 (Isomorphism Criterion for $TLb_2$). Two algebras $L(z)$ and $L(z')$ from $TLb_2$ are isomorphic if and only if there exist complex numbers $A_0$, $A_1$, and $B_1$: $A_0B_1 \neq 0$, and the following conditions hold:

$$b_{0,0}' = \frac{A_0^2b_{0,0} + A_1A_0b_{0,1} + A_1^2b_{1,1}}{A_0B_1},$$

$$b_{0,1}' = \frac{A_0b_{0,1} + 2A_1b_{1,1}}{A_0^3},$$

$$b_{1,1}' = \frac{B_1b_{1,1}}{A_0^3},$$

$$b_{1,2}' = \frac{B_1b_{1,2}}{A_0^3}.$$

Proof. Part “If”. Let $L_1$ and $L_2$ from $TLb_2$ be isomorphic: $f : L_1 \cong L_2$. We choose the corresponding adapted bases $\{e_0, e_1, e_2, e_3, e_4\}$ and $\{e'_0, e'_1, e'_2, e'_3, e'_4\}$ in $L_1$ and $L_2$, respectively. Then in these bases the algebras will be presented as $L(z)$ and $L(z')$.

According to Proposition 4.1 one has

$$e'_0 = f(e_0) = A_0e_0 + A_1e_1 + A_2e_2 + A_3e_3 + A_4e_4,$n

$$e'_1 = f(e_1) = B_1e_1 + B_2e_2 + B_3e_3 + B_4e_4.$$

Then we get

$$e'_2 = f(e_2) = [f(e_1), f(e_0)] = A_0B_1e_2 + A_0B_2e_3 + (A_0B_3 + A_1B_1b_{1,1} + (A_2B_1 - A_1B_2)b_{1,2}) e_4,$n

$$e'_3 = f(e_3) = [f(e_2), f(e_0)] = A_0^2B_1e_3 + (A_2^3B_2 - A_0A_1B_1b_{1,2})e_4,$n

$$e'_4 = f(e_4) = [f(e_3), f(e_0)] = A_0^3B_1e_4.$$

By using the table of multiplications, one finds the relation between the coefficients $b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}$ and $b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2}$. First consider the equality $[f(e_0), f(e_0)] = b'_{0,0}f(e_4)$, we get Eq. (1) and from the equality $[f(e_1), f(e_0)] + [f(e_0), f(e_1)] = b'_{0,1}f(e_4)$, we have (2) and $[f(e_1), f(e_1)] = b'_{1,1}f(e_4)$ gives (3). Finally the equality (4) comes out from $[f(e_1), f(e_2)] = b'_{1,2}f(e_4)$.

“Only if” part.

Let Eqs. (1)-(4) hold. Then the base change (5) above is adapted, and it transforms $L(z)$ into $L(z')$. Indeed,

$$[e'_0, e'_0] = \left[ \sum_{i=0}^{4} A_i e_i, \sum_{i=0}^{4} A_i e_i \right]$$

$$= A_0^2[e_0, e_0] + A_0A_1[e_0, e_1] + A_0A_1[e_1, e_0] + A_1^2[e_1, e_1]$$

$$= (A_0^3b_{0,0} + A_0A_1b_{0,1} + A_1^2b_{1,1})e_4 = b'_{0,0}A_0^3B_1e_4 = b'_{0,0}e'_4.$$
\[ [e'_0, e'_1] = \sum_{i=0}^{4} A_i e_i + \sum_{j=1}^{4} B_j e_j \]

\[ = -(A_4 B_1 e_2 + A_0 B_2 e_3 + (A_1 B_1 b_{1,1} + A_2 B_1 b_{1,2} - A_1 B_2 b_{1,2} + A_0 B_3) e_4) \]

\[ + B_1 (B_{0,1} A_0 + 2 A_1 B_{1,1}) e_4 \]

\[ = -e'_2 + A_4 B_1 b'_{0,1} e_4 = -e'_2 + b'_{0,1} e'_4. \]

Using the same manner, one can prove that \([e'_1, e'_2] = b'_{1,1} e'_4\) and \([e'_1, e'_3] = b'_{1,2} e'_n.\]

Now we list all isomorphism classes of algebras from \(TLb_5\).

Represent \(TLb_5\) as a disjoint union of the following subsets:

\[ U_1 = \{ L(x) \in TLb_5 : b_{1,1} \neq 0, b_{1,2} \neq 0 \}, \]

\[ U_2 = \{ L(x) \in TLb_5 : b_{1,1} \neq 0, b_{1,2} = 0, \Delta \neq 0 \}, \]

\[ U_3 = \{ L(x) \in TLb_5 : b_{1,1} \neq 0, b_{1,2} = \Delta = 0 \}, \]

\[ U_4 = \{ L(x) \in TLb_5 : b_{1,1} = 0, b_{0,1} \neq 0, b_{1,2} \neq 0 \}, \]

\[ U_5 = \{ L(x) \in TLb_5 : b_{1,1} = 0, b_{0,1} \neq 0, b_{1,2} = 0 \}, \]

\[ U_6 = \{ L(x) \in TLb_5 : b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,2} \neq 0 \}, \]

\[ U_7 = \{ L(x) \in TLb_5 : b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,2} = 0 \}, \]

\[ U_8 = \{ L(x) \in TLb_5 : b_{1,1} = b_{0,1} = b_{0,0} = 0, b_{1,2} \neq 0 \}. \]

\[ U_9 = \{ L(x) \in TLb_5 : b_{1,1} = b_{0,1} = b_{0,0} = b_{1,2} = 0 \}. \]

**Proposition 5.11.**

1. Two algebras \(L(x)\) and \(L(x')\) from \(U_1\) are isomorphic if and only if \(\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta.\)

2. For any \(\lambda\) from \(C\), there exists \(L(x) \in U_1 : \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta = \lambda.\)

**Proof.** 1. “\(\Rightarrow\)” Let \(L(x)\) and \(L(x')\) be isomorphic. Then due to Theorem 3.1, it is easy to see that \(\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta.\)

“\(\Leftarrow\)” Let the equality \(\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta\) hold. Consider the base change (5) above with \(A_0 = \frac{b_{1,1}}{b_{1,2}},\ A_1 = -\frac{b_{1,1}}{2b_{1,2}},\ and\ \ B_1 = \frac{b_{1,1}}{b_{1,2}}.\ This\ changing\ leads\ \(L(x)\) into \(L\left(\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta, 0, 1, 1\right).\ An\ analogous\ base\ change\ transforms\ \(L(x')\) into \(L\left(\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta', 0, 1, 1\right).\)

Since \(\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta\) then \(L(x)\) is isomorphic to \(L(x').\)

2. Obvious. □
Proposition 5.12. The subsets $U_2, U_3, U_4, U_5, U_6, U_7, U_8$, and $U_9$ are single orbits with the representatives $L(1, 0, 1, 0), L(0, 0, 1, 0), L(0, 1, 0, 1), L(0, 1, 0, 0), L(1, 0, 0, 0), L(1, 0, 1, 0), L(0, 0, 0, 1)$, and $L(0, 0, 0, 0)$, respectively.

Theorem 5.6. Let $L \in Tlb_5$. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras:

1) $L(0, \lambda, \lambda)$:

$$L^l = \{[e_i, e_0] = e_i, 1 \leq i \leq 3, [e_0, e_i] = -e_{i+1}, 2 \leq i \leq 3, [e_0, e_0] = e_4, [e_0, e_1] = -e_2, [e_1, e_1] = \lambda e_4, [e_2, e_1] = -[e_1, e_2] = \lambda e_4, \lambda \in \mathbb{C}.\$$

2) $L(0, 1, 0)$:

$$L^l, [e_0, e_0] = e_4, [e_0, e_1] = -e_2, [e_1, e_1] = e_4.\$$

3) $L(1, 0, 1)$:

$$L^l, [e_0, e_1] = -e_2 + e_4, [e_0, e_0] = e_4, [e_2, e_1] = [e_1, e_2] = e_4.\$$

4) $L(1, 0, 0)$:

$$L^l, [e_0, e_1] = -e_2 + e_4, [e_0, e_0] = e_4.\$$

5) $L(2, 1, 1)$:

$$L^l, [e_0, e_0] = e_4, [e_0, e_1] = -e_2 + 4e_4, [e_1, e_1] = 2e_4, [e_2, e_1] = -[e_1, e_2] = e_4.\$$

6) $L(2, 1, 0)$:

$$L^l, [e_0, e_0] = e_4, [e_0, e_1] = -e_2 + 4e_4, [e_1, e_1] = 2e_4.\$$

7) $L(0, 0, 1)$:

$$L^l, [e_0, e_0] = e_4, [e_2, e_1] = -[e_1, e_2] = e_4, [e_0, e_1] = -e_2.\$$

5.3.2. 6-Dimensional case. $Tlb_6$ can be represented by the following table of multiplication:

$$TLb_6 = \left\{ \begin{array}{ll}
[e_i, e_0] = e_{i+1}, & 1 \leq i \leq 4, \\
[e_0, e_i] = -e_{i+1}, & 2 \leq i \leq 4, \\
[e_0, e_0] = b_{0,0}e_5, [e_0, e_1] = -e_2 + b_{0,1}e_5, [e_1, e_1] = b_{1,1}e_5, \\
[e_1, e_2] = -[e_2, e_1] = b_{1,2}e_4 + b_{1,3}e_5, \\
[e_1, e_3] = -[e_3, e_1] = b_{1,3}e_5, \\
[e_1, e_4] = -[e_4, e_1] = -[e_2, e_3] = [e_3, e_2] = -b_{2,3}e_6. & \\
\end{array} \right.$$
Elements of $TLb_6$ will be denoted by $L(z)$, where $z = (b_{0,1}, b_{1,1}, b_{1,2}, b_{1,3}, b_{2,3})$.

**Theorem 5.7** (Isomorphism Criterion for $TLb_6$). Two filiform Leibniz algebras $L(x)$ and $L(x')$ from $TLb_6$ are isomorphic iff there exist $A_0, A_1, B_1, B_2, B_3 \in \mathbb{C}$ such that $A_0B_1(A_0 + A_1b_{2,3}) \neq 0$, and the following equalities hold:

\[
\begin{align*}
\tau_{0,0} & = \frac{A_0^2b_{0,0} + A_0A_1b_{0,1} + A_1^2b_{1,1}}{A_0^3b_1(A_0 + A_1b_{2,3})}, \\
\tau_{0,1} & = \frac{A_0b_{0,1} + 2A_1b_{1,1}}{A_0(A_0 + A_1b_{2,3})}, \\
\tau_{1,1} & = \frac{B_1b_{1,1}}{A_0(A_0 + A_1b_{2,3})}, \\
\tau_{1,2} & = \frac{B_1^2b_{1,2}}{A_0^2}, \\
\tau_{1,3} & = 2A_0A_1B_2^2b_{1,2}^2 + A_0^2B_1^2b_{1,3} + (A_0^3(-2B_1B_3 + B_2^2) + A_1^3B_1^2b_{1,2}^2) b_{2,3}, \\
\tau_{2,3} & = \frac{B_1b_{2,3}}{A_0 + A_1b_{2,3}}.
\end{align*}
\]

The proof is similar to that of Theorem 3.1.

Represent $TLb_6$ as a union of the following subsets:

\[
\begin{align*}
U_1 & = \{ L(x) \in TLb_6 : b_{2,3} \neq 0, b_{1,1} \neq 0 \}, \\
U_2 & = \{ L(x) \in TLb_6 : b_{2,3} \neq 0, b_{1,1} = 0, b_{0,1} \neq 0 \}, \\
U_3 & = \{ L(x) \in TLb_6 : b_{2,3} \neq 0, b_{1,1} = b_{0,1} = 0, b_{1,2} \neq 0, b_{0,0} \neq 0 \}, \\
U_4 & = \{ L(x) \in TLb_6 : b_{2,3} \neq 0, b_{1,1} = b_{0,1} = 0, b_{1,2} \neq 0, b_{0,0} = 0 \}, \\
U_5 & = \{ L(x) \in TLb_6 : b_{2,3} \neq 0, b_{1,1} = 0, b_{1,2} = 0, b_{0,0} \neq 0 \}, \\
U_6 & = \{ L(x) \in TLb_6 : b_{2,3} \neq 0, b_{1,1} = b_{0,1} = b_{1,2} = b_{0,0} = 0 \}, \\
U_7 & = \{ L(x) \in TLb_6 : b_{2,3} = 0, b_{1,2} \neq 0, b_{1,1} \neq 0 \}, \\
U_8 & = \{ L(x) \in TLb_6 : b_{2,3} = 0, b_{1,2} \neq 0, b_{1,1} = 0, b_{0,1} \neq 0 \}, \\
U_9 & = \{ L(x) \in TLb_6 : b_{2,3} = 0, b_{1,2} \neq 0, b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0 \}, \\
U_{10} & = \{ L(x) \in TLb_6 : b_{2,3} = 0, b_{1,2} \neq 0, b_{1,1} = b_{0,1} = b_{0,0} = 0 \}, \\
U_{11} & = \{ L(x) \in TLb_6 : b_{2,3} = b_{1,1} = 0, b_{1,2} \neq 0, b_{1,1} \neq 0 \}, \\
U_{12} & = \{ L(x) \in TLb_6 : b_{2,3} = b_{1,2} = 0, b_{1,1} \neq 0, b_{1,3} = 0, \Delta \neq 0 \}, \\
U_{13} & = \{ L(x) \in TLb_6 : b_{2,3} = b_{1,2} = 0, b_{1,1} \neq 0, b_{1,3} = 0, \Delta = 0 \}, \\
U_{14} & = \{ L(x) \in TLb_6 : b_{2,3} = b_{1,2} = b_{1,1} = 0, b_{0,1} \neq 0, b_{1,3} \neq 0 \}, \\
U_{15} & = \{ L(x) \in TLb_6 : b_{2,3} = b_{1,2} = 0, b_{0,0} \neq 0, b_{1,3} \neq 0 \}.
\end{align*}
\]
Proposition 5.13.

1. Two algebras \( L(x) \) and \( L(x') \) from \( U_1 \) are isomorphic if and only if

\[
\left( \frac{b_{2,3}'}{2b_{1,1}' - b_{0,1}'b_{2,3}'} \right)^2 \Delta' = \left( \frac{b_{2,3}}{2b_{1,1} - b_{0,1}b_{2,3}} \right)^2 \Delta
\]

and

\[
\frac{(2b_{1,1}' - b_{2,3}'b_{0,1})^3 b_{1,2}'^3}{b_{2,3}'^3 b_{1,1}^4} = \frac{(2b_{1,1} - b_{2,3}b_{0,1})^3 b_{1,2}^3}{b_{2,3}^3 b_{1,1}^4}.
\]

2. For any \( \lambda_1, \lambda_2 \in \mathbb{C} \), there exists \( L(x) \in U_1 \):

\[
\left( \frac{b_{2,3}}{2b_{1,1} - b_{0,1}b_{2,3}} \right)^2 \Delta = \lambda_1, \quad \frac{(2b_{1,1} - b_{2,3}b_{0,1})^3 b_{1,2}^3}{b_{2,3}^3 b_{1,1}^4} = \lambda_2.
\]

Then orbits from the set \( U_1 \) can be parameterized as \( L(\lambda_1, 0, 1, \lambda_2, 0, 1) \), \( \lambda_1, \lambda_2 \in \mathbb{C} \).

Proposition 5.14.

1. Two algebras \( L(x) \) and \( L(x') \) from \( U_2 \) are isomorphic if and only if

\[
\frac{(b_{1,1}' - b_{2,3}'b_{0,1})^4 b_{1,2}'^3}{b_{2,3}'^3 b_{0,1}^4} = \frac{(b_{0,1} - b_{2,3}b_{0,1})^4 b_{1,2}^3}{b_{2,3}^3 b_{0,1}^4}.
\]

2. For any \( \lambda \in \mathbb{C} \), there exists \( L(x) \in U_2 \):

\[
\frac{(b_{0,1} - b_{2,3}b_{0,1})^4 b_{1,2}^3}{b_{2,3}^3 b_{0,1}^4} = \lambda.
\]

Therefore, orbits from \( U_2 \) can be parameterized as \( L(0, 1, 0, \lambda, 0, 1), \lambda \in \mathbb{C} \).

Proposition 5.15.

1. Two algebras \( L(x) \) and \( L(x') \) from \( U_7 \) are isomorphic if and only if

\[
\frac{4b_{0,1}b_{1,2}^4 - 2b_{1,3}b_{0,1}b_{1,2}^2 + b_{1,3}^2b_{1,1}}{b_{1,2}^2b_{1,1}} = \frac{4b_{0,1}b_{1,2}^4 - 2b_{1,3}b_{0,1}b_{1,2}^2 + b_{1,3}^2b_{1,1}}{b_{1,2}^2b_{1,1}}
\]

and

\[
\frac{(b_{0,1}b_{1,2}^2 - b_{1,3}b_{1,1})}{b_{1,2}b_{1,1}} = \frac{(b_{0,1}b_{1,2}^2 - b_{1,3}b_{1,1})}{b_{1,2}b_{1,1}}.
\]
2. For any \( \lambda_1, \lambda_2 \in \mathbb{C} \), there exists \( L(\lambda) \in U_7 \):

\[
\frac{4b_{0,0}b_{1,2}^4 - 2b_{1,3}b_{0,1}b_{1,2}^2 + b_{1,3}^3b_{1,1}}{b_{1,2}b_{1,1}^2} = \lambda_1, \quad \frac{(b_{0,1}b_{1,2}^2 - b_{1,3}b_{1,1})^2}{b_{1,2}b_{1,1}^2} = \lambda_2.
\]

Orbits from \( U_7 \) can be parameterized as \( L(\lambda_1, \lambda_2, 1, 1, 0, 0), \lambda_1, \lambda_2 \in \mathbb{C} \).

**Proposition 5.16.**

1. Two algebras \( L(\lambda) \) and \( L(\lambda') \) from \( U_8 \) are isomorphic if and only if

\[
\frac{(2b_{0,0}b_{1,2}^2 - b_{1,3}b_{0,1})^3}{b_{1,2}b_{0,1}^3} = \frac{(2b_{0,0}b_{1,2}^2 - b_{1,3}b_{0,1})^3}{b_{1,2}b_{0,1}^3}.
\]

2. For any \( \lambda \in \mathbb{C} \), there exists \( L(\lambda) \in U_8 \):

\[
\frac{(2b_{0,0}b_{1,2}^2 - b_{1,3}b_{0,1})^3}{b_{1,2}b_{0,1}^3} = \lambda.
\]

The orbits from the set \( U_8 \) can be parameterized as \( L(\lambda, 1, 0, 1, 0, 0), \lambda \in \mathbb{C} \).

**Proposition 5.17.**

1. Two algebras \( L(\lambda) \) and \( L(\lambda') \) from \( U_{11} \) are isomorphic if and only if

\[
\left( \frac{b_{1,3}}{b_{1,1}} \right)^6 = \left( \frac{b_{1,3}}{b_{1,1}} \right)^6.
\]

2. For any \( \lambda \in \mathbb{C} \), there exists \( L(\lambda) \in U_{11} \):

\[
\left( \frac{b_{1,3}}{b_{1,1}} \right)^6 = \lambda.
\]

The orbits from \( U_{11} \) can be parameterized as \( L(\lambda, 0, 1, 0, 1, 0), \lambda \in \mathbb{C} \).

**Proposition 5.18.** The subsets \( U_3, U_4, U_5, U_6, U_9, U_{10}, U_{12}, U_{13}, U_{14}, U_{15}, U_{16}, U_{18}, \) and \( U_{19} \) are single orbits with the representatives \( \lambda(1, 0, 0, 1, 0, 1), \lambda(0, 0, 0, 1, 0, 1), \lambda(1, 0, 0, 0, 0, 1), \lambda(0, 0, 0, 0, 0, 1), \lambda(1, 0, 0, 0, 0, 1), \lambda(1, 0, 0, 0, 1, 0), L(1, 0, 0, 1, 0, 0), L(0, 0, 0, 1, 0, 0) \), \( L(1, 0, 0, 1, 0, 0), L(0, 0, 0, 1, 0, 0), L(1, 0, 0, 1, 0, 0), L(0, 0, 0, 1, 0, 0), L(1, 0, 0, 1, 0, 0), L(0, 0, 0, 1, 0, 0), L(0, 0, 0, 1, 0, 0), L(0, 0, 0, 0, 0), \) and \( L(0, 0, 0, 0, 0, 0) \), respectively.

**Theorem 5.8.** Let \( L \) be a complex filiform Leibniz algebra in \( T\mathbb{L}_6 \). Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras:

1) \( L(0, \lambda_1, \lambda_2) \):

\[
L(0, \lambda_1, \lambda_2) = \{ [e_i, e_0] = e_{i+1}, 1 \leq i \leq 4, [e_0, e_i] = -e_{i+1}, 2 \leq i \leq 4 \},
\]
\[ [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2, \quad [e_1, e_1] = \lambda_i e_5, \quad [e_2, e_1] = -[e_1, e_2] = \lambda e_4, \]
\[ = \lambda_2 e_4 + \lambda_1 e_5, \quad [e_3, e_1] = -[e_1, e_3] = \lambda_2 e_5, \quad \lambda_1, \lambda_2 \in \mathbb{C}, \quad \lambda_1 \neq 0. \]

2) \( L(0, \lambda, \lambda, 0) : \)
\[ L_0, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2, \quad [e_2, e_1] = -[e_1, e_2] = \lambda e_4, \]
\[ [e_1, e_1] = [e_3, e_1] = -[e_1, e_3] = \lambda e_5, \quad \lambda \in \mathbb{C}. \]

3) \( L(0, 1, 0, 0) : \)
\[ L_0, \quad [e_0, e_0] = [e_1, e_1] = e_5, \quad [e_0, e_1] = -e_2. \]

4) \( L(\lambda, 0, \lambda, 0) : \)
\[ L_0, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + \lambda e_5, \quad [e_2, e_1] = -[e_1, e_2] = \lambda e_4, \]
\[ [e_3, e_1] = -[e_1, e_3] = \lambda e_5, \quad \lambda \in \mathbb{C}^*. \]

5) \( L(1, 0, 1, -2) : \)
\[ L_0, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + e_5, \quad [e_2, e_1] = -[e_1, e_2] = e_4 - 2e_5, \]
\[ [e_3, e_1] = -[e_1, e_3] = e_5. \]

6) \( L(1, 0, 0, 1) : \)
\[ L_0, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + e_5, \quad [e_2, e_1] = -[e_1, e_2] = e_5. \]

7) \( L(1, 0, 0, 0) : \)
\[ L_0, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + e_5, \]

8) \( L(\lambda, \lambda^2, \lambda, 0) : \)
\[ L_0, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + \lambda e_5, \quad [e_1, e_1] = \lambda^2 e_5, \]
\[ [e_2, e_1] = -[e_1, e_2] = \lambda e_4, \quad [e_3, e_1] = -[e_1, e_3] = \lambda e_5, \quad \lambda \in \mathbb{C}^*. \]

9) \( L(4, 4, 0, 1) : \)
\[ L_0, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + 4e_5, \quad [e_1, e_1] = 4e_5, \quad [e_2, e_1] \]
\[ = -[e_1, e_2] = e_5. \]

10) \( L(4, 4, 0, 0) : \)
\[ L_0, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + 4e_5, \quad [e_1, e_1] = 4e_5. \]
11) $L(0, 0, 1, 0)$:

$$L'_6, \quad [e_0, e_0] = [e_3, e_1] = -[e_1, e_1] = e_5, \quad [e_0, e_1] = -e_2, \quad [e_2, e_1] = -[e_1, e_2] = e_4.$$ 

12) $L(0, 0, 0, 1)$:

$$L'_6, \quad [e_0, e_0] = [e_1, e_1] = -[e_1, e_2] = e_5, \quad [e_0, e_1] = -e_2.$$ 

As we have mentioned before the class $TLb_n$ contains $n$-dimensional filiform Lie algebras. The sets $U_5, U_9$ in the 5-dimensional case and the sets $U_4, U_6, U_{10}, U_{18}, U_{19}$ in the 6-dimensional case represent Lie cases. Our classification here agreed with the classification of 5- and 6-dimensional filiform Lie algebras in Gómez et al. [9].

CONCLUSION

The methods and algorithms of this article are applicable to any fixed dimensional case. They have been implemented in dimensions at most 9, and complete lists of all isomorphism types of algebras from $Tb_n$ ($n = 5, 6, 7, 8, 9$) are obtained.

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