A Class of Nonassociative Algebras Including Flexible and Alternative Algebras, Operads and Deformations

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Abstract

There exist two types of nonassociative algebras whose associator satisfies a symmetric relation associated with a 1-dimensional invariant vector space with respect to the natural action of the symmetric group \( \Sigma_3 \). The first one corresponds to the Lie-admissible algebras and this class has been studied in a previous paper of Remm and Goze. Here we are interested by the second one corresponding to the third power associative algebras.

Keywords: Nonassociative algebras; Alternative algebras; Third power associative algebras; Operads

Introduction

Recently, we have classified for binary algebras, Cf. [1], relations of nonassociativity which are invariant with respect to an action of the symmetric group on three elements \( \Sigma_3 \). In particular, we have investigated two classes of nonassociative algebras, the first one corresponds to algebras whose associator \( A_1 \) satisfies

\[ A_1 = (\text{Id} - r_{12} - r_{13} + c + c^3) = 0, \]  

and the second

\[ A_2 = (\text{Id} + r_{12} + r_{13} + c + c^3) = 0, \]  

where \( r_i \) denotes the transposition exchanging \( i \) and \( j \), \( c \) is the 3-cycle \((1,2,3)\).

These relations are in correspondence with the only two irreducible one-dimensional subspaces of \( K[\Sigma_3] \) with respect to the action of \( \Sigma_3 \) which are invariant with respect to an action of the symmetric group on three elements \( \Sigma_3 \). To fix notation we define

\[ G_i = \{ | \}, G_2 = \{ t_2 \}, G_3 = \{ t_3 \}, G_6 = \{ t_2, t_3 \}, G_3 = \{ c \}, G_4 = \Sigma_3, \]

where \( \prec \) is the cyclic group subgroup generated by \( \sigma \). To each subgroup \( G_i \) we associate the vector \( v_i \) of \( K[\Sigma_3] \):

\[ v_i = \sum \sigma. \]

Lemma 1. The one-dimensional subspace \( K[v_i] \) of \( K[\Sigma_3] \) generated by 

\[ v_i = v_j = \sum_{\sigma \in \Sigma_3} \sigma \]

is an irreducible invariant subspace of \( K[\Sigma_3] \) with respect to the right action of \( \Sigma_3 \) on \( K[\Sigma_3] \).

Recall that there exists only one two-dimensional invariant subspace of \( K[\Sigma_3] \), the second being generated by the vector \( \sum_{\sigma \in \Sigma_3} \delta(\sigma) \sigma \) where \( \delta(\sigma) \) is the sign of \( \sigma \). As we have precise in the introduction, this case has been studied in literature of Remm [1].

Definition 2. A \( G_i \)-\( p^i \)-associative algebra is a \( K \)-algebra \((A, \mu)\) whose associator

\[ A = \mu^*(\mu \otimes \text{Id} - \text{Id} \otimes \mu) \]

satisfies

\[ A = \Phi_{G_i}^i \otimes \Phi_{G_i}^i = 0, \]

where \( \Phi_{G_i}^i : A^{p^i} \rightarrow A^{p^i} \) is the linear map

\[ \Phi_{G_i}^i (x_1 \otimes x_2 \otimes x_3) = \sum_{\sigma \in \Sigma_3} (x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}). \]

Let \( O(v_i) \) be the orbit of \( v_i \) with respect to the right action

\[ \Sigma_3 \times K[\Sigma_3] \rightarrow K[\Sigma_3] \]

\[ (\sigma, [\sigma]_A \sigma) \rightarrow \sum_{\sigma \in \Sigma_3} [\sigma]_A \sigma \sigma^{-1} \sigma_\sigma \]

Then putting \( F_{G_i} = K(O(v_i)) \) we have

\[ \dim F_{G_1} = 6, \]

\[ \dim F_{G_2} = 6, \]

\[ \dim F_{G_3} = 3, \]

\[ \dim F_{G_4} = 2, \]

\[ \dim F_{G_5} = 1. \]

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Proposition 3. Every $G_i$-p$^3$-associative algebra is third power associative.

Recall that a third power associative algebra is an algebra $(A, \mu)$ whose assiciator satisfies $A_3(x, x, x) = 0$. Linearizing this relation, we obtain

$$A_3(x, x, y) = 0,$$

since each of the invariant spaces $F_i$ contains the vector $v_{x,y}$. We deduce the proposition.

Remark. An important class of third power associative algebras is the class of power associative algebras, that is, algebras such that any element generates an associative subalgebra.

What are $G_i$-p$^3$-associative algebras?

(1) If $i = 1$, since $v_{x,y} = 1d$, the class of $G_i$-p$^3$-associative algebras is the full class of associative algebras.

(2) If $i = 2$, the associator of a $G_i$-p$^3$-associative algebra $A$ satisfies

$$A_2(x, x, y) + A_2(x, y, x) + A_2(y, x, x) = 0,$$

and this is equivalent to

$$A_2(x, x, y),$$

for all $x, y \in A$.

(3) If $i = 3$, the associator of a $G_i$-p$^3$-associative algebra $A$ satisfies

$$A_3(x, x, y),$$

that is,

$$A_3(x, y, y),$$

for all $x, y \in A$.

Sometimes $G_i$-p$^3$-associative algebras are called left-alternative algebras, $G_i$-p$^3$-associative algebras are right-alternative algebras. An alternative algebra is an algebra which satisfies the $G_2$ and $G_i$-p$^3$-associativity.

(4) If $i = 4$, we have $A_4(x, y, y, x)$ for all $x, y \in A$, and the class of $G_i$-p$^3$-associative algebras is the class of flexible algebras.

(5) If $i = 5$, the class of $G_i$-p$^3$-associative algebras corresponds to the class of $G_i$-associative algebras [2].

(6) If $i = 6$, the associator of a $G_i$-p$^3$-associative algebra satisfies

$$A_6(x, y, x),$$

and the skew-symmetric product $[x, y] = \mu(x, y) - \mu(y, x)$, then the $G_i$-p$^3$-associative identity is equivalent to

$$[x \ast y, z] + [y \ast z, x] + [z \ast x, y] = 0.$$

Definition 4. A $(\{,\}, \ast)$-admissible-algebra is a $K$-vector space $A$ provided with two multiplications:

(a) a symmetric multiplication * ,

(b) a skew-symmetric multiplication $[,]$ satisfying the identity

$$[x \ast y, z] + [y \ast z, x] + [z \ast x, y] = 0$$

for any $x, y \in A$.

Then a $G_i$-p$^3$-associative algebra can be defined as $(\{,\}, \ast)$-admissible algebra.

Remark: Poisson algebras. A $K$-Poisson algebra is a vector space $P$ provided with two multiplications, an associative commutative one $x \cdot y$ and a Lie bracket $[x, y]$, which satisfy the Leibniz identity

$$[x \cdot y, z] - x \cdot [y, z] - [x, z] \cdot y = 0.$$

In studies of Remm [7], it is shown that these conditions are equivalent to provide $P$ with a nonassociative multiplication $x \cdot y$ satisfying

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z - x \cdot (y \cdot z - y \cdot (x \cdot z)) = 0.$$

If we denote by $A^i(x, y, z) = x \cdot (y \cdot z)$ and $A^i(x, y, z) = (x \cdot y) \cdot z$ then the previous identity is equivalent to

$$\frac{1}{5} A^i(x, y, z) - A^i(x, y, z) - A^i(x, y, z) = 0,$$

where $w_i = 3d$ and $w_i = 3d + 2i - 1$. In fact the class of Poisson algebras is a subclass of a family of nonassociative algebras defined by conditions on the associator. The product satisfies

$$A^i(x, y, z) + A^i(x, y, z) = A^i(z, y, x) = 0,$$

and

$$A^i(x, y, z) + A^i(x, y, z) = 0,$$

so it is a subclass of the class of algebras which are flexible and $G_i$-p$^3$-associative [1].

The Operads $G_i$-p$^3$-Ass and their Dual

For each $i \in \{1, \ldots, 6\}$, the operads for $G_i$-p$^3$-associative algebras will be denoted by $G_i$-p$^3$-Ass. The operads $\{G_i$-p$^3$-Ass$\}_{i=1}^6$ are binary quadratic operads, that is, operads of the form $\mathcal{P} = \Gamma(E)/\langle R \rangle$, where $\Gamma(E)$ denotes the free operad generated by a $\Sigma$-module $E$ placed in arity 2 and $\langle R \rangle$ is the operadic ideal generated by a $\Sigma$-invariant subspace $R \subset \Gamma(E)$. Then the dual operad $\mathcal{P}^!$ is the quadratic operad $\mathcal{P}^! = \Gamma(E)/\langle R^! \rangle$, where $R^! \subset \Gamma(E)$ is the annihilator of $R \subset \Gamma(E)$ in the pairing

$$\langle x, y \rangle \cdot \langle x, y \rangle = 0,$$

where $x, y \in E$. For the general notions of binary quadratic operads [4,5], recall that a quadratic operad $\mathcal{P}$ is Koszul if the free $\mathcal{P}$-algebra based on a $K$-vector space $V$ is Koszul for any vector space $V$. This property is preserved by duality and can be studied using generating functions of $\mathcal{P}$ and of $\mathcal{P}^!$ [4,6]. Before studying the Koszulness of the operads $G_i$-p$^3$-Ass, we will compute the homology of an associative algebra which will be useful to look if $G_i$-p$^3$-Ass are Koszul or not.

Let $A$ be the two-dimensional associative algebra given in a basis $\{e_1, e_2\}$ by $e_1 e_2 = e_1, e_2 e_1 = e_2, e_2 e_2 = 0$. Recall that the Hochschild homology of an associative algebra is given by the complex $(C_n(A), d_n)$ where $C_n(A, A) = A \otimes A^n$ and the differentials $d_n : C_{n+1}(A, A) \to C_n(A, A)$ are given by

$$d_n c = \sum_{i=1}^{n+1} (-1)^i c_{i-1} c_i.$$

where $c$ is a homogeneous element of $C_n(A, A)$.
We deduce that \(G_{r} \cdot p^1 \text{Ass} \) is the maximal current operad of \(P\) defined in [7,8].

The operad \((G_{r} \cdot p^1 \text{Ass})\)

The operad \((G_{r} \cdot p^1 \text{Ass})\) is the operad for left-alternative algebras. It is the quadratic operad \(P = \Gamma(E)/(R), \) where the \(\Sigma_{i}\)-invariant subspace \(R\) of \(\Gamma(E)(3)\) is generated by the vectors

\[
(x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3) + (x_1 \cdot x_2) \cdot x_3 - x_1 \cdot x_2 \cdot x_3.
\]

The annihilator \(R^2\) of \(R\) with respect to the pairing \((3)\) is generated by the vectors

\[
\begin{aligned}
(x_1 \cdot x_2) \cdot x_3 - x_1 \cdot (x_2 \cdot x_3) = 0 \\
(x_1 \cdot x_2) \cdot x_3 - (x_1 \cdot x_2) \cdot x_3 = 0
\end{aligned}
\]

We deduce from direct calculations that \(\dim R^2 = 9\) and

**Proposition 5.** The \((G_{r} \cdot p^1 \text{Ass})\)-algebras are associative algebras satisfying

\[
abc = -bac.
\]

Recall that \((G_{r} \text{Ass})\)-algebras are associative algebras satisfying

\[
abc = bac.
\]

and this operad is classically denoted \(\text{Perm}\).

**Theorem 6.** The operad \((G_{r} \cdot p^1 \text{Ass})\) is not Koszul [9].

*Proof.* It is easy to describe \((G_{r} \cdot p^1 \text{Ass})\) for any \(n\). In fact \((G_{r} \cdot p^1 \text{Ass})\) corresponds to associative elements satisfying

\[
x_1 x_2 x_3 x_4 = -x_2 x_3 x_1 x_4 = -x_5 (x_1 x_4) x_3 = x_1 x_2 x_3 x_4 = -x_5 x_3 x_2 x_4
\]

and \((G_{r} \cdot p^1 \text{Ass})\) \((4)\) is \([0]\). Let \(P\) be \((G_{r} \cdot p^1 \text{Ass})\). The generating function of \(P = (G_{r} \cdot p^1 \text{Ass})\) is

\[
g_{P}(x) = \frac{1}{1+2x} = x + x^2 + \frac{x^3}{2}
\]

But the generating function of \(P = (G_{r} \cdot p^1 \text{Ass})\) is

\[
g_{P}(x) = x + x^2 + \frac{5}{2} x^3 + O(x^4)
\]

and if \((G_{r} \cdot p^1 \text{Ass})\) is Koszul, then the generating functions should be related by the functional equation

\[
g_P(-g_{P}(-x)) = x
\]

and it is not the case so both \((G_{r} \cdot p^1 \text{Ass})\) and \((G_{r} \cdot p^1 \text{Ass})\) are not Koszul.

By definition, a quadratic operad \(P\) is Koszul if any free \(P\)-algebra on a vector space \(V\) is a Koszul algebra. Let us describe the free algebra \(F_{(G_{r} \cdot p^1 \text{Ass})}(V)\) when \(\dim V = 1\) and 2.

A \((G_{r} \cdot p^1 \text{Ass})\)-algebra \(A\) is an associative algebra satisfying

\[
xyz = -yxx,
\]

for any \(x, y, z \in A\). This implies \(yzt = 0\) for any \(x, y, z \in A\). In particular we have

\[
\begin{aligned}
x & = 0, \\
y & = 0,
\end{aligned}
\]

for any \(x, y \in A\). If \(\dim V = 1\), \(F_{(G_{r} \cdot p^1 \text{Ass})}(V)\) is of dimension 2 and given by

\[
\begin{aligned}
e_1 e_1 & = e_2, \\
e_1 e_3 & = e_2 e_1 = e_3 e_1 = 0.
\end{aligned}
\]

In fact if \(V = \mathbb{K} [e_1]\) then in \(F_{(G_{r} \cdot p^1 \text{Ass})}(V)\) we have \(e_1 = 0\). We deduce that \(F_{(G_{r} \cdot p^1 \text{Ass})}(V) = A\) and \(F_{(G_{r} \cdot p^1 \text{Ass})}(V) = A\) is not Koszul. It is easy to generalize this construction. If \(\dim V = n\), then \(\dim F_{(G_{r} \cdot p^1 \text{Ass})}(V) = \frac{n^2 + n + 2}{2}\) and if \(\{e_i\}_{i=1,\ldots,n}\) is a basis of \(V\) then \(\{e_i, e_i, e_i, e_i e_i, e_i e_i e_i\}\) for \(i, j = 1, \ldots,n\) and \(l, m, p = 1, \ldots,n\) with \(m > i\), is a basis of \(F_{(G_{r} \cdot p^1 \text{Ass})}(V)\). For example, if \(n = 2\), the basis of \(F_{(G_{r} \cdot p^1 \text{Ass})}(V)\) is

\[
\{e_1 = e_2, e_1 = e_3, e_1 = e_4, e_1 = e_2 e_1, e_1 = e_2 e_3, e_1 = e_2 e_4, e_2 = e_3 e_2, e_2 = e_4 e_2\}
\]

and the multiplication table is

\[
\begin{array}{cccccccc}
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
v_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
v_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

For this algebra we have
The relation $G_{ss}$ is the linear span of $d_1(v_1,v_2) = v_3 - v_6$, $d_1(v_1,v_6) = v_7 - d_1(v_1,v_3) = d_1(v_2,v_3)$, $d_1(v_2,v_6) = v_8 - d_1(v_1,v_7) = d_1(v_1,v_4)$, and $\text{Ker} \ d_1$ is of dim 64. The space $\text{Im} \ d_1$ doesn’t contain in particular the vectors $(v_i, v_i)$ for $i = 1, 2$ because these vectors $v_i$ are not in the derived subalgebra. Since these vectors are in $\text{Ker} \ d_1$ we deduce that the second space of homology is not trivial.

**Proposition 7.** The current operad of $G_{ss}$pAss is $G_{ss}$pAss = Perm.

This is directly deduced of the definition of the current operad [7].

**The operad $(G_{ss})pAss$**

It is defined by the module of relations generated by the vector $(x_1, x_2) \alpha - (x_1, x_2) \alpha + (x_1, x_2) \alpha - (x_1, x_2) \alpha$, and $R^+$ is the linear span of

$$(x_1, x_2) \alpha - (x_1, x_2) \alpha + (x_1, x_2) \alpha - (x_1, x_2) \alpha,$$

and $$(x_1, x_2) \alpha + (x_1, x_2) \alpha.$$ 

**Proposition 8.** A $(G_{ss})pAss$-algebra is an associative algebra $A$ satisfying

$$abc = -acb,$$

for any $a, b, c \in A$.

Since $(G_{ss})pAss$ is basically isomorphic to $(G_{ss})pAss$ we deduce that $(G_{ss})pAss$ is not Koszul.

**The operad $(G_{ss})pAss$**

Remark that a $(G_{ss})pAss$-algebra is generally called flexible algebra. The relation $A_x(x_1, x_2, x_3) + A_x(x_1, x_2, x_3) = 0$ is equivalent to $A_x(x, y, x) = 0$ and this denotes the flexibility of $(A, \mu)$.

**Proposition 9.** A $(G_{ss})pAss$-algebra is an associative algebra satisfying

$$abc = -cba.$$

This implies that $\dim \ (G_{ss})pAss (3) = 3$. We have also $x_{ss}(x_{ss}x_{ss}x_{ss}) = (-1)^{ss} x_{ss}x_{ss}x_{ss}$ for any $\sigma \in \Sigma_3$. This gives $\dim \ (G_{ss})pAss (4) = 1$. Similarly

$x_{ss}(x_{ss}x_{ss}x_{ss}) = -x_{ss}(x_{ss}x_{ss}x_{ss}) = x_{ss}(x_{ss}x_{ss}x_{ss}) = -x_{ss}(x_{ss}x_{ss}x_{ss}) = x_{ss}(x_{ss}x_{ss}x_{ss}) = -x_{ss}(x_{ss}x_{ss}x_{ss})$ (the algebra is associative so we put some parenthesis just to explain how we pass from one expression to another). We deduce $(G_{ss})pAss (5) = 0$ and more generally $(G_{ss})pAss (a) = 0$ for $a \geq 5$.

The generating function of $(G_{ss})pAss$ is

$$f(x) = x + x^3 + \frac{x^5}{2} + \frac{x^7}{12}.$$ 

Let $F_{ss}(V) = \mathcal{F}(V)$ be the free $(G_{ss})pAss$-algebra based on the vector space $V$. In this algebra we have the relations

$$a^3 = 0,$$

for any $a, b \in V$. Assume that $\dim V = 1$. If $[e_i]$ is a basis of $V$, then

$$e_i^3 = 0$$

and $F_{ss}(V) = \mathcal{F}(V)$ is not a Koszul algebra.

**Proposition 10.** The operad for flexible algebra is not Koszul.

Let us note that, if $\dim V = 2$ and $[e_i, e_j]$ is a basis of $V$, then $F_{ss}(V)$ is generated by $\{ e_i, e_j, e_i^2, e_i e_j, e_j e_i, e_j^2, e_i e_j e_i, e_j e_i e_j \}$ and is of dimension 12.

**Proposition 11.** We have

$$(G_{ss})pAss = (G_{ss})-Ass.$$ 

This means that a $(G_{ss})pAss$ is an associative algebra $A$ satisfying $abc = cba$, for any $a, b, c \in A$.

**The operad $(G_{ss})pAss$**

It coincides with $(G_{ss})Ass$ and this last has been studied by Remm [2].

**The operad $(G_{ss})pAss$**

A $(G_{ss})pAss$-algebra $(A, \mu)$ satisfies the relation

$$A_x(x_1, x_2, x_3) + A_x(x_1, x_2, x_3) + A_x(x_1, x_2, x_3) = 0.$$ 

The dual operad $(G_{ss})pAss^\ast$ is generated by the relations

$$\left\{ \begin{array}{l} \{ x_{ss}(x_1, x_2) = x_{ss}(x_1, x_2), \\
\{ x_{ss}(x_1, x_2) = (-1)^{ss} x_{ss}(x_{ss}, x_{ss}) x_{ss}, \end{array} \right.$$ 

for all $\sigma \in \Sigma_3$.

We deduce

**Proposition 12.** A $(G_{ss})pAss$-algebra is an associative algebra $A$ which satisfies

$$abc = -bac = -cba = -bac = cba,$$

for any $a, b, c \in A$. In particular

$$a^3 = 0,$$

for any $a, b, c \in A$. In particular

$$\left\{ \begin{array}{l} a^3 = 0, \\
abc = ab = cba = 0. \end{array} \right.$$ 

**Lemma 13.** The operad $(G_{ss})pAss$ satisfies $(G_{ss})pAss (4) = 0$.

**Proof.** We have in $(G_{ss})pAss (4)$ that

$$x_{ss}(x_1, x_2) = x_{ss}(x_1, x_2),$$

so $x_{ss}x_{ss}x_{ss} = 0$. We deduce that the generating function of $(G_{ss})pAss (4)$ is

$$f'(x) = x + x^3 + \frac{x^5}{6}.$$ 

If this operad is Koszul the generating function of the operad $(G_{ss})pAss$ should be of the form

$$f(x) = x + x^3 + \frac{11}{6} x^5 + \frac{25}{6} x^7 + \frac{127}{12} x^9 + \ldots$$
Then (isomorphic to), of the, and of the residual field is isomorphic to $A$ such that, if $G_{\mathcal{P}}$ algebras, that is, by the operad (deformations cohomology differ [12]. is based on the minimal model of $\mathcal{P}$ operad is the "standard"-cohomology called the operadic cohomology. If the parametrizes deformations. Since the field of rational fractions. This case corresponds to the classical Gerstenhaber deformations. Since $\mathbb{K}$ non archimedian extension field of $\mathbb{K}$, with a valuation $\mu$ satisfies $\mathbb{K}$-valued deformation can be viewed as a $\mathbb{K}$ valued operad, we have that $\mu: \mathbb{K} \to \mathbb{K}$ and also by the classical $\mathbb{K}$-formal series. In this case $\mathcal{P} = \{ \sum_{a \in K} a \}, \mathbb{K} = \mathbb{K}(\{\})$ the field of rational fractions. This case corresponds to the classical Gerstenhaber deformations. Since $A$ is a local ring, all the notions of valued deformations coincide [11]. We know that there exists always a cohomology which parametrizes deformations. If the operad $\mathcal{P}$ is Koszul, this cohomology is the "standard"-cohomology called the operadic cohomology. If the operad $\mathcal{P}$ is not Koszul, the cohomology which governs deformations is based on the minimal model of $\mathcal{P}$ and the operadic cohomology and deformations cohomology differ [12].

In this section we are interested by the case of left-alternative algebras, that is, by the operad $(G_{\mathcal{P}}, \mathcal{P}, \mathcal{A})$ and also by the classical alternative algebras.

**Deformations and cohomology of left-alternative algebras**

A $\mathbb{K}$-left-alternative algebra $(A, \mu)$ is a $\mathbb{K}$-free algebra. Then $\mu$ satisfies
\[
A_{\mu}(x_1, x_2, x_3, x_4) + A_{\mu}(x_2, x_3, x_1, x_4) = 0.
\]
A valued deformation can be viewed as a $\mathbb{K}[[t]]$-algebra $(A \otimes \mathbb{K}[[t]], \mu)$ whose product $\mu_\mu$ is given by
\[
\mu_\mu = \mu + \sum_{i=1}^{t} \nu_i.
\]

The operadic cohomology: It is the standard cohomology $H^*(G_{\mathcal{P}}, \mathcal{A})$ of the $(G_{\mathcal{P}}, \mathcal{P}, \mathcal{A})$-algebra $(A, \mu)$. It is associated to the cochains complex
\[
\cdots \rightarrow C^{p}_\mu(A, \mathcal{A}) \rightarrow C^{p+1}_\mu(A, \mathcal{A}) \rightarrow \cdots
\]
where $\mathcal{P} = (G_{\mathcal{P}}, \mathcal{P}, \mathcal{A})$ and
\[
H^p(A, \mathcal{A}) = \operatorname{Hom}(\mathcal{P}(p) \otimes \mathcal{A}^{p}, \mathcal{A}).
\]
Since $(G_{\mathcal{P}}, \mathcal{P}, \mathcal{A})(4) = 0$, we deduce that
\[
H^p(A, \mathcal{A}) = 0 \text{ for } p \geq 4.
\]

because the cochain complex is a short sequence
\[
\cdots \rightarrow C^{p}_\mu(A, \mathcal{A}) \rightarrow C^{p+1}_\mu(A, \mathcal{A}) \rightarrow C^{p+2}_\mu(A, \mathcal{A}) \rightarrow \cdots
\]
and the boundary operator are given by
\[
\delta f(a, b) = f(ab) - f(a) + f(ab),
\]
\[
\delta^2 f(a, b, c) = \phi(ab, c) + \phi(ba, c) - \phi(ab, bc) - \phi(b, ac)
\]
\[
\phi(a, b, c) = \phi(ab, c) - \phi(a, bc) + \phi(ba, c) - \phi(b, ac).
\]

The deformations cohomology: The minimal model of $(G_{\mathcal{P}}, \mathcal{P}, \mathcal{A})$ is a homology isomorphism
\[
\delta \mu_\mu \rightarrow (\Gamma(E), \delta)
\]
of dg-operads such that the image of $\delta$ consists of decomposable elements of the free operad $\Gamma(E)$. Since $(G_{\mathcal{P}}, \mathcal{P}, \mathcal{A})(1) = \mathbb{K}$, this minimal model exists and it is unique. The deformations cohomology $H'(A, \mathcal{A})_{\delta\delta}$ of $\mathcal{A}$ is the cohomology of the complex
\[
\cdots \rightarrow C^p_{\delta\delta}(A, \mathcal{A}) \rightarrow C^{p+1}_{\delta\delta}(A, \mathcal{A}) \rightarrow \cdots
\]
where
\[
\begin{align*}
C^0_{\delta\delta}(A, \mathcal{A}) &= \operatorname{Hom}(A, \mathcal{A}), \\
C^1_{\delta\delta}(A, \mathcal{A}) &= \operatorname{Hom}(\mathcal{P}(1) \otimes \mathcal{A}, A).
\end{align*}
\]

The Euler characteristic of $E(q)$ can be read off from the inverse of the generating function of the operad $(G_{\mathcal{P}}, \mathcal{P}, \mathcal{A})$
\[
g_{\mathcal{P}, \mathcal{A}}(1) = t + t^2 + \frac{5}{2} t^3 + \frac{13}{2} t^4 + \frac{53}{12} t^5
\]
which is
\[
y(t) = t + t^2 + \frac{3}{2} t^3 + \frac{5}{2} t^4 + O(t^5).
\]

We obtain in particular
\[
\gamma(E(4)) = 0.
\]
Each one of the modules $E(p)$ is a graded module $(E(p))$ and
\[
\gamma(E(p)) = \dim E_p(p) - \dim E_{p+1}(p) - \dim E_{p+2}(p) + \cdots
\]
We deduce:
\begin{itemize}
  \item $E(2)$ is generated by two degree 0 bilinear operations $\mu_2: V \cdot V \to V$,
  \item $E(3)$ is generated by three degree 1 trilinear operations $V^3 \to V$,
  \item $E(4) = 0$.
\end{itemize}

Considering the action of $\Sigma_p$ on $E(n)$ we deduce that $E(2)$ is generated by a binary operation of degree 0 whose differential satisfies
\[
\delta(\mu_2) = 0,
\]
\[
E(3) \text{ is generated by a trilinear operation of degree one such that}
\]
\[
\delta(\mu_3) = \mu_3 + \mu_2 \cdot \mu_1 - \mu_2 \cdot \mu_1 - \mu_3 \cdot \mu_1 + \mu_3 \cdot \mu_2 - \mu_3 \cdot \mu_2 - \mu_1 \\
(\mu_2 \cdot \mu_1) - (\mu_2 \cdot \mu_1)
\]
we have $\mu_3 \cdot \mu_2 - (\mu_2 \cdot \mu_1) = (b(a)c)$. Since $E(4) = 0$ we deduce

**Proposition 15.** The cohomology $H'(A, \mathcal{A})_{\delta\delta}$ which governs deformations of right-alternative algebras is associated to the complex
\[
\cdots \rightarrow C^p_{\delta\delta}(A, \mathcal{A}) \rightarrow C^{p+1}_{\delta\delta}(A, \mathcal{A}) \rightarrow \cdots
\]
with
\[
H^p(A, \mathcal{A}) = 0 \text{ for } p \geq 4.
\]
Alternative algebras

Recall that an alternative algebra is given by the relation

\[ A_p(x_1, x_2, x_3) = 0. \]

Theorem 16. An algebra \( (A, \mu) \) is alternative if and only if the associator satisfies

\[ \Phi^1 = 0, \]

with \( v = 2Id + \tau_3 + \tau_2 + c_1 \).

Proof. The associator satisfies \( A_p = \Phi^1 \) with \( v = \mu + \tau_3 \) and \( v = Id + \tau_2 \). The invariant subspace of \( \mathbb{K}[\Sigma] \) generated by \( v_1 \) and \( v_2 \) is of dimension 5 and contains the vector \( \sum_{i=0}^{3} \sigma_i \). From literature of Remm [1], the space is generated by the orbit of the vector \( v \).

Proposition 17. Let \( Alt \) be the operad for alternative algebras. Its dual is the operad for associative algebras satisfying

\[ abc - bac = cba - ach + bca + cab = 0. \]

Remark. The current operad \( \tilde{Alt} \) is the operad for associative algebras satisfying \( abc = bac = cba = abc = bea \), that is, 3-commutative algebras so \( \tilde{Alt} = LieAdm \).

In literature of Dzhumadil’daev and Zusmanovich [9], one gives the generating functions of \( P = Alt \) and \( P' = Alt' \)

\[ g_p(x) = x + \frac{2}{3} x^4 + \frac{7}{3} x^3 + \frac{32}{41} x^2 + \frac{175}{31} x + \frac{180}{61} + O(x^7), \]

\[ g_p(x) = x + \frac{2}{3} x^4 + \frac{7}{3} x^3 + \frac{32}{41} x^2 + \frac{175}{31} x + \frac{180}{61} + O(x^7), \]

and conclude to the non-Koszulness of \( Alt \).

The operadic cohomology is the cohomology associated to the complex

\[ C_p(Alt(p) \otimes \mathcal{A}, \mathcal{A}), \delta_p). \]

Since \( Alt(p) \) \( p \geq 6 \), we deduce the short sequence

\[ C_p(Alt(p), \delta_p) \rightarrow C_p(Alt(p), \delta_p) \rightarrow \cdots \rightarrow C_p(Alt(p), \delta_p) \rightarrow 0. \]

But if we compute the formal inverse of the function \( -g_{\mu}(-x) \) we obtain

\[ x + \frac{5}{6} x^2 + \frac{11}{12} x + O(x^3). \]

Because of the minus sign it can not be the generating function of the operad \( P' = Alt' \). So this implies also that both operad are not Koszul. But it gives also some information on the deformation cohomology. In fact if \( \Gamma(E) \) is the free operad associated to the minimal model, then

\[ \dim \chi(E(2)) = -2, \]

\[ \dim \chi(E(3)) = -5, \]

\[ \dim \chi(E(4)) = -12, \]

\[ \dim \chi(E(5)) = -15, \]

\[ \dim \chi(E(6)) = +110. \]

Since \( \chi(E(6)) = \sum (-1)^i \dim E_i(6) \), the graded space \( E(6) \) is not concentrated in degree even. Then the 6-cochains of the deformation cohomology are 6-linear maps of odd degree.

References


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