Elementary Analysis Math 140B—Winter 2007
Sample Midterm; February 28, 2007

NOTE:

THE MIDTERM WILL CONSIST OF 4 PROBLEMS, SIMILAR TO THE FOLLOWING PROBLEMS.

1. Suppose that f is strictly increasing on $[a, b]$.

   (a) Show that f is one-to-one.

   (b) Show that f^{-1} is strictly increasing.

2. Let f and g be continuous functions on $[a, b]$ and differentiable on (a, b), with $g(a) \neq g(b)$. Prove that there exists a point $x_0 \in (a, b)$ such that
$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}.$$
(Hint: Consider the function $[f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$.)

3. If f is differentiable on (a, b) and if $f'(x) \neq 0$ for all x in the interval, prove that f is monotonic on (a, b).

4. Show that the function $f(x) = x^3 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$ has a derivative for all values of x and that f' is continuous at $x = 0$ but not differentiable there.

5. Let $f(x) = x/(1 + e^{1/x})$ for $x \neq 0$ and $f(0) = 0$. Does $f'(0)$ exist?

6. Let f be a function which is defined for all x with the properties (i) $f(a + b) = f(a)f(b)$, (ii) $f(0) = 1$, (iii) f is differentiable at $x = 0$. Show that f is differentiable for all values of x.

7. Define the function f by $f(2^{-n}) = 2^{-2n}$ for $n = 1, 2, \ldots$ and $f(x) = 0$ for other values of x. Is f differentiable at $f = 0$?

8. Suppose that f is continuous on (a, b) and that f is known to be differentiable on the interval except possibly at the point $x_0 \in (a, b)$. Suppose further that $\lim_{x \to x_0} f'(x)$ exists. Use the Mean Value Theorem to prove that f is differentiable at x_0 and that f' is continuous at x_0.