NOTE:

THE MIDTERM WILL CONSIST OF 4 PROBLEMS, SIMILAR TO THE FOLLOWING PROBLEMS.

1. Suppose that $f$ is strictly increasing on $[a, b]$.
   
   (a) Show that $f$ is one-to-one.
   
   (b) Show that $f^{-1}$ is strictly increasing.

2. Let $f$ and $g$ be continuous functions on $[a, b]$ and differentiable on $(a, b)$, with $g(a) \neq g(b)$.
   
   Prove that there exists a point $x_0 \in (a, b)$ such that
   
   $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$$
   
   (Hint: Consider the function $[f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$)

3. If $f$ is differentiable on $(a, b)$ and if $f'(x) \neq 0$ for all $x$ in the interval, prove that $f$ is monotonic on $(a, b)$.

4. Show that the function $f(x) = x^3 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$ has a derivative for all values of $x$ and that $f'$ is continuous at $x = 0$ but not differentiable there.

5. Let $f(x) = x/(1 + e^{1/x})$ for $x \neq 0$ and $f(0) = 0$. Does $f'(0)$ exist?

6. Let $f$ be a function which is defined for all $x$ with the properties (i) $f(a + b) = f(a)f(b)$, (ii) $f(0) = 1$, (iii) $f$ is differentiable at $x = 0$. Show that $f$ is differentiable for all values of $x$.

7. Define the function $f$ by $f(2^{-n}) = 2^{-2n}$ for $n = 1, 2, \ldots$ and $f(x) = 0$ for other values of $x$. Is $f$ differentiable at $f = 0$?

8. Suppose that $f$ is continuous on $(a, b)$ and that $f$ is known to be differentiable on the interval except possibly at the point $x_0 \in (a, b)$. Suppose further that $\lim_{x \to x_0} f'(x)$ exists. Use the Mean Value Theorem to prove that $f$ is differentiable at $x_0$ and that $f''$ is continuous at $x_0$. 