Complex Analysis Math 147—Winter 2006
Review Problems—Chapters 1-3; January 30, 2006

1. Write the number in the form $a + bi$ with $a, b \in \mathbf{R}$.
   $\frac{3}{i} + \frac{i}{3}$, $(2+i)(-1-i)(3-2i)$.

2. Let z be a complex number such that $\Re z > 0$. Prove that $\Re(1/z) > 0$.

3. Evaluate $3i^{11} + 6i^3 + \frac{8}{i^{20}} + i^{-1}$

4. Describe the set of points in the complex plane that satisfies each of the following
   $\Im z = 2$, $|2z - i| = 4$, $|z| = \Re z + 2$, $|z| = 3|z-1|$, $|z - i| < 2$.

5. Prove that if $(\overline{z})^2 = z^2$, then z is either real or purely imaginary.

6. Decide which of the following statements are true.
   (a) $\operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$ if $z_1 \neq 0$ and $z_2 \neq 0$.
   (b) $\operatorname{Arg} \overline{z} = -\operatorname{Arg} z$ if z is not a real number.

7. Write each of the given numbers in the form $a + bi$ with $a, b \in \mathbf{R}$.
   $\exp(-i\pi/4)$, $\frac{\exp(1+i3\pi)}{\exp(-1+i\pi/2)}$, $\exp(\exp i)$

8. Write each of the given numbers in the polar form $r\cos\theta + ir\sin\theta$.
   $\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right)^3$, $\frac{2+2i}{-\sqrt{3}+i}$, $\frac{2i}{3\exp(4+i)}$

9. The function $\gamma(t) = \exp(it)$, $0 \leq t \leq 2\pi$ describes the unit circle traversed in the counter-clockwise direction. Describe each of the following curves.
   (a) $\gamma(t) = 3\exp(it)$, $0 \leq t \leq 2\pi$
   (b) $\gamma(t) = 2\exp(it) + i$, $0 \leq t \leq 2\pi$

10. Use the identity $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ for any positive integer n to show that $(\sqrt{3} - i)^7 = -64\sqrt{3} + i64$.

11. Write each of the following functions in the form $w = u(x, y) + iv(x, y)$.
    (a) $f(z) = 3z^2 + 5z + i + 1$
    (b) $h(z) = \frac{z+i}{z^2+1}$
    (c) $F(z) = \exp(3z)$

12. Show that the function $\operatorname{Arg} z$ is discontinuous at each point of the non-positive real axis.

13. Prove that if f is continuous at $z = z_0$, then so are the functions $\overline{f(z)}$ and $\Re f(z)$.

14. Find each of the following limits.

    (a) $\lim_{z\to 2\pi i}(\exp z - \exp(-z))$

    (b) $\lim_{z\to -\pi i}(z+1)\exp\left(\frac{z^2+\pi^2}{z+\pi i}\right)$

15. Where are each of the following functions analytic?

    (a) $8\overline{z}+i$

    (b) $\frac{z^3+2z+i}{z-5}$

    (c) $x^2+y^2+y-2+ix$

    (d) $|z|^2+2z$

16. Use the Cauchy-Riemann equations to determine where the following functions are differentiable.

    (a) $f(x+iy) = (x^{4/3}y^{5/3} + ix^{5/3}y^{4/3})/(x^2+y^2)$ for $z \neq 0$ and $f(0) = 0$.

    (b) $g(x+iy) = 3x^2+2x-3y^2-1+i(6xy+2y)$.

17. Show that if f is analytic and real-valued on a connected open set, then f is a constant function.

18. Suppose $f(z)$ and $\overline{f(z)}$ are analytic on an open connected set. Show that f is a constant function.

19. Write each of the following numbers in the form $a+bi$.

    (a) $\exp(2+i\pi/4)$

    (b) $\cos(1-i)$

20. Show that the formula $\exp(iz) = \cos z + i\sin z$ holds for all complex z.

21. Evaluate each of the following.

    (a) $\log i$

    (b) $\operatorname{Log}(\sqrt{3}+i)$

22. Find all values of the following.

    (a) $2^{\pi i}$

    (b) $(1+i)^3$

23. Find the principal value of the following.

    (a) i^{2i}

    (b) $(1+i)^{1-i}$

24. Find all solutions of $\sin z = 2$.