1. Write the number in the form $a + bi$ with $a, b \in \mathbb{R}$.

$$\frac{3}{i} + \frac{i}{3} = -8i/3 \quad , \quad (2 + i)(-1 - i)(3 - 2i) = -1 - 3i.$$ 

2. Let $z$ be a complex number such that $\Re z > 0$. Prove that $\Re(1/z) > 0$.

$1/z = 1/(x + iy) = (x - iy)/(x^2 + y^2)$ so $\Re(1/z) = x/(x^2 + y^2) > 0$.

3. Evaluate

$$3i^{11} + 6i^3 + 8i^{20} + i^{-1} = 8 - 10i$$

4. Describe the set of points in the complex plane that satisfies each of the following

- $\Im z = 2$ is the line $y = 2$
- $|2z - i| = 4$ is the circle with center $i/2$ and radius 2
- $|z| = \Re z + 2$ is the parabola $y^2 = 4(x + 1)$
- $|z| = 3|z - 1|$ is the circle with center $9/8$ and radius $3/8$
- $|z - i| < 2$ is the inside of a circle with center $i$ and radius 2.

5. Prove that if $(z)^2 = z^2$, then $z$ is either real or purely imaginary.

$$(x - iy)^2 = (x + iy)2 \Rightarrow x^2 - 2ixy - y^2 = x^2 + 2ixy - y^2 \Rightarrow xy = 0$$

6. Decide which of the following statements are true.

- (a) $\text{Arg} z_1 z_2 = \text{Arg} z_1 + \text{Arg} z_2$ if $z_1 \neq 0$ and $z_2 \neq 0$. False
- (b) $\text{Arg} z = -\text{Arg} z$ if $z$ is not a real number. True

7. Write each of the given numbers in the form $a + bi$ with $a, b \in \mathbb{R}$.

$$\exp(-i\pi/4) = (1/\sqrt{2})(1 - i) \quad , \quad \frac{\exp(1+i3\pi)}{\exp(-1+i\pi/2)} = e^2i$$

$$\exp(\exp i) = e^{\cos 1}(\cos(\sin 1) + i \sin(\sin 1))$$

8. Write each of the given numbers in the polar form $r \cos \theta + ir \sin \theta$.

$$\left(\cos \frac{2\pi}{9} + i \sin \frac{2\pi}{9}\right)^3 = \exp(i2\pi/3) \text{ or } \exp(i2\pi/3 + 2\pi k), \; k \in \mathbb{Z}$$

$$\frac{2+2i}{\sqrt{3}+i} = 2\sqrt{2} \exp(i\pi/4) \text{ or } 2\sqrt{2} \exp(i\pi/4 + 2\pi k), \; k \in \mathbb{Z}$$

$$\frac{2i}{3\exp(4+i)} = (2/3e^4) \exp(-i\pi/2) \text{ or } (2/3e^4) \exp(-i\pi/2 + 2\pi k), \; k \in \mathbb{Z}$$

9. The function $\gamma(t) = \exp(it)$, $0 \leq t \leq 2\pi$ describes the unit circle traversed in the counter-clockwise direction. Describe each of the following curves.

- (a) $\gamma(t) = 3 \exp(it)$, $0 \leq t \leq 2\pi$ is a circle with center 0 and radius 3 traversed in the counterclockwise direction once.
- (b) $\gamma(t) = 2 \exp(it) + i$, $0 \leq t \leq 2\pi$ is a circle with center $i$ and radius 2 traversed in the counterclockwise direction once.
10. Use the identity \( |r(\cos \theta + i \sin \theta)|^n = r^n(\cos n\theta + i \sin n\theta) \) for any positive integer \( n \) to show that \((\sqrt{3} - i)^7 = -64\sqrt{3} + 64i\).

\[
(\sqrt{3} - i)^7 = (2 \exp(-i\pi/6))^7 = 2^7 \exp(-7\pi/6) = 2^7(\sqrt{3}/2 + i/2) = -64\sqrt{3} + 64i
\]

11. Write each of the following functions in the form \( w = u(x, y) + iv(x, y) \).

(a) \( f(z) = 3z^2 + 5z + i + 1 = 3(x^2 - y^2) + 5x + 1 + i(6xy + 5y + 1) \)

(b) \( h(z) = \frac{z+i}{z+i+1} = x/(x^2 + (y-1)^2) + i(1-y)/(x^2 + (y-1)^2) \)

(c) \( F(z) = \exp(3z) = e^{3x} \cos 3y + ie^{3x} \sin 3y \)

12. Show that the function \( \text{Arg} z \) is discontinuous at each point of the non-positive real axis.

First of all, the function \( \text{Arg} z \) is not defined for \( z = 0 \). Let \( z_0 = x_0 \) be a negative real number. If \( y > 0 \), then \( \text{Arg}(x_0 + iy) = \pi - \tan^{-1}(y/x_0) \to \pi \) as \( y \to 0 \).

Also, if \( y < 0 \), then \( \text{Arg}(x_0+iy) = -\pi - \tan^{-1}(y/x_0) \to -\pi \) as \( y \to 0 \). Therefore, \( \lim_{z \to x_0} \text{Arg} z \) does not exist, and so \( \text{Arg} z \) is not continuous at \( z_0 = x_0 \) if \( x_0 < 0 \).

13. Prove that if \( f \) is continuous at \( z = z_0 \), then so are the functions \( \overline{f(z)} \) and \( \Re f(z) \).

\[
|\overline{f(z)} - \overline{f(z_0)}| = |f(z) - f(z_0)| < \epsilon \text{ if } |z - z_0| < \delta
\]

\[
|\Re f(z) - \Re f(z_0)| = |(f(z) + \overline{f(z)})/2 - (f(z_0) + \overline{f(z_0)})/2| \leq \frac{1}{2} |f(z) - f(z_0)| + \frac{1}{2} |\overline{f(z)} - \overline{f(z_0)}|
\]

14. Find each of the following limits.

(a) \( \lim_{z \to 2\pi i}(\exp z - \exp(-z)) = 0 \)

(b) \( \lim_{z \to -\pi i}(z + 1) \exp \left(\frac{z^2 + \pi^2}{z + \pi i}\right) = 1 - \pi i \)

15. Where are each of the following functions analytic?

(a) \( 8\pi + i \) NOWHERE

(b) \( \frac{z^3+2z+i}{z-5} \) ON \( C - \{5\} \)

(c) \( x^2 + y^2 + y - 2 + ix \) NOWHERE

(d) \( |z|^2 + 2z \) NOWHERE

16. Use the Cauchy-Riemann equations to determine where the following functions are differentiable.

(a) \( f(x + iy) = (x^{4/3}y^{5/3} + ix^{5/3}y^{4/3})/(x^2 + y^2) \) for \( z \neq 0 \) and \( f(0) = 0 \). THIS PROBLEM IS TOO MESSY—PLEASE IGNORE IT!

(b) \( g(x + iy) = 3x^2 + 2x - 3y^2 - 1 + i(6xy + 2y) \). EVERYWHERE
17. Show that if \( f \) is analytic and real-valued on a connected open set, then \( f \) is a constant function.

\[ f = u + iv \text{ and } v = 0, \text{ so } u_x = v_y = 0 \text{ and } u_y = v_x = 0. \text{ Since we are on an open connected set, we have } u \text{ is a constant and so } f \text{ is constant.} \]

18. Suppose \( f(z) \) and \( f'(z) \) are analytic on an open connected set. Show that \( f \) is a constant function.

\[ f = u + iv \text{ analytic implies } u_x = v_y \text{ and } u_y = -v_x. \text{ If } f = u - iv \text{ analytic implies } u_x = -v_y \text{ and } u_y = -(v_x) = v_x. \text{ Then } u_x = u_y = 0 \text{ and } u \text{ is a constant since the set is open and connected. Same for } v. \]

19. Write each of the following numbers in the form \( a + bi \).

(a) \( \exp(2 + i\pi/4) = e^2/\sqrt{2}(1+i) \)
(b) \( \cos(1 - i) = \cos 1 \cosh 1 + i \sin 1 \sinh 1 \)

20. Show that the formula \( \exp(iz) = \cos z + i \sin z \) holds for all complex \( z \).

\[ \exp(iz) = \exp(-y + ix) = e^{-y} \cos x + ie^{-y} \sin x \text{ and} \]
\[ \cos z + i \sin z = \cos x \cosh y - i \sin x \sinh y + i \sin x \cosh y - \cos x \sinh y \]
\[ = \cos x (\cosh y - \sinh y) + i \sin x (\cosh y - \sinh y) = (\cos x)e^{-y} + i(\sin x)e^{-y} \]

21. Evaluate each of the following.

(a) \( \log i = \{i(\pi/2 + 2\pi k) : k \in \mathbb{Z}\} \)
(b) \( \Log (\sqrt{3} + i) = \log \sqrt{2} + i\pi/6 \)

22. Find all values of the following.

(a) \( 2^{\pi i} = \{\exp(\pi i \Log 2 - 2\pi^2 k) : k \in \mathbb{Z}\} \)
(b) \( (1 + i)^3 = 2(i - 1) \)

23. Find the principal value of the following.

(a) \( i^{2i} = e^{-\pi} \)
(b) \( (1 + i)^{1-i} = \sqrt{2}e^{\pi/4} \exp(i(\pi/4 - \log \sqrt{2})) \)

24. Find all solutions of \( \sin z = 2 \).

If \( z = x + iy \) then \( \sin z = 2 \) if and only if \( \sin x \cosh y = 2 \) and \( \cos x \sinh y = 0 \).

This leads to \( z = \pi/2 + 2k\pi + i \cosh^{-1}(2) \) for any \( k \in \mathbb{Z} \).