Complex Analysis Math 147—Winter 2008
Review Problems for final exam; March 14, 2008

1. Let $f$ be entire and suppose that the second derivative of $f$ is bounded: $|f''(z)| \leq M$ for all $z \in \mathbb{C}$. Prove that $f$ is a polynomial of degree at most 2.

   Hint: Use Liouville’s theorem on $f''$ and then the fact that in a polygonally connected open set, an analytic function whose derivative vanishes everywhere must be a constant.

2. Suppose that $f$ is entire and that $|f(z)| \leq |z|^{2.5}$ for all $|z| > 100$.
   (a) Given $z_0$, show that for $R$ sufficiently large, $|f^{(3)}(z_0)| \leq 3!R^{3.5}/(R - |z_0|)^4$.
   (b) Prove that $f$ must be a polynomial of degree at most 2.

   Hint: Use the Cauchy integral formula for the third derivative of $f$ and estimate the integral using the fact that since $|z| = R$, $|z - z_0| \geq R - |z_0|$

3. Let $f$ be an entire function and let $a, b \in \mathbb{C}$.
   (a) Evaluate the integral $\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} \, dz$ if $R > |a|$ and $R > |b|$.
   (b) Use the result of (a) to give a proof of Liouville’s theorem.

   Hint: Estimate the integral and let $R \to \infty$.

4. (a) Show that the following function is analytic on the open unit disk: $f(z) = \int_0^1 \frac{dt}{1-tz}$

   Hint: Use Morera’s theorem and an interchange of the order of integration.

   (b) Find a power series expansion for this function.

   Hint: Use the known power series for the integrand and interchange the summation and integration.

5. Let $0 < r < R$ and $A := \{z \in \mathbb{C} : r \leq |z| \leq R\}$. Show that there is a positive number $\epsilon$ such that for each polynomial $p$,

   $\sup \{|p(z) - z^{-1}| : z \in A\} \geq \epsilon$.

   Hint: Suppose it is not true, either find a sequence of polynomials converging uniformly to $1/z$ on compact subsets of $A$, then integrate over a circle, or alternatively, use the estimate on contour integrals.

6. Let $f_n$ be a sequence of functions which are continuous on the closed unit disk $\{|z| \leq 1\}$ and analytic on the open disk $\{|z| < 1\}$. Suppose the $f_n$ converges uniformly to a function $f$ on the unit circle $\{|z| = 1\}$. Show that $f$ can be extended to a function $g$ on $\{|z| \leq 1\}$ which is analytic on $\{|z| < 1\}$.

   Hint: Apply the maximum modulus theorem to $f_n - f_m$.  

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7. (a) Prove that if \( f \) is an automorphism of the open unit disk (that is, \( f : \{ |z| < 1 \} \to \{ |z| < 1 \} \) is analytic, one-to-one and onto) and if \( f(0) = 0 \), then \( f(z) = e^{i\theta}z \) for all \( |z| < 1 \).

Hint: Assuming, as you may, that \( f^{-1} \) is analytic, use Schwarz’s lemma for \( f \) and \( f^{-1} \).

For (b) and (c), let \( D \) be an open subset of \( \mathbb{C} \) and fix a point \( a \in D \).

(b) Show that there is at most one analytic function \( f : D \to \{ |z| < 1 \} \) which is one-to-one and onto and satisfies \( f(a) = 0 \) and \( f'(a) > 0 \).

Hint: If \( f \) and \( g \) are two such functions, then \( f \circ g^{-1} \) is an automorphism of the open unit disk. (Also, recall that \( (g^{-1})'(g(z)) = 1/g'(z) \).

(c) Let \( g \) be any one-to-one analytic function mapping \( D \) onto the open unit disk. Show that

\[
g(z) = \frac{e^{i\theta}f(z) + \alpha}{1 + \overline{\alpha}e^{i\theta}f(z)} \quad (z \in D)
\]

where \( \alpha \) and \( \theta \) are defined by \( \alpha = g(a) \) and \( g'(a) = e^{i\theta}|g'(a)| \).

Hint: Let \( h := \varphi_{-\alpha} \circ g \) and evaluate \( h'(a) \), where as usual \( \varphi_{\alpha}(z) = \frac{z+\alpha}{1+\overline{\alpha}z} \).

8. Let \( f \) be analytic on an open polygonally connected set \( D \) containing an open interval \( I \) of the real axis. Suppose that \( D \) is symmetric about the real axis: \( z \in D \iff \overline{z} \in D \), and that \( f \) is real-valued on \( I \). Prove that \( \overline{f(z)} = f(z) \) for all \( z \in D \).

Hint: Define \( g(z) = \overline{f(z)} \). Show, by using the Cauchy-Riemann equations that \( g \) is analytic on \( D \). Then use the identity theorem to show that \( f = g \).

9. Show that there does not exist a one-to-one analytic function of \( \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) onto the open unit disk \( \{ z \in \mathbb{C} : |z| < 1 \} \).

Hint: If \( f \) is such a function, show first that it would have an analytic extension \( g \) to \( z = 0 \). Then consider the cases \( |g(0)| = 1 \) and \( |g(0)| < 1 \). In the latter case there exists \( \beta \) with \( 0 < |\beta| < 1 \) and \( f(\beta) = g(0) \). (Note that \( f(\beta) = g(\beta) \).) Obtain a contradiction by justifying the following steps:

- For \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( f^{-1}(B(g(0), \delta)) \subset B(\beta, \epsilon) \)
- For this \( \delta \), there exists \( \delta' > 0 \) such that \( g(B(0, \delta')) \subset B(g(\beta), \delta) \) and hence \( f(B(0, \delta') - \{0\}) \subset B(g(\beta), \delta) \)
- \( B(0, \delta') - \{0\} \subset B(\beta, \epsilon) \)

10. Suppose \( f \) is analytic and zero-free in a simply connected domain \( D \). Show there is an analytic function \( g \) in \( D \) with \( e^{g(z)} = f(z) \).

Hint: Fix \( z_0 \in D \) and define \( h(z) = \frac{1}{z_0} \int_{z_0}^{z} \frac{f'(w)}{f(z)} \, dw \) where the integral is taken over any curve in \( D \) from \( z_0 \) to \( z \). Explain why \( h \) is well-defined and analytic in \( D \) with derivative \( h' = f'/f \) and then show that \( \frac{d}{dz}(e^{-h(z)}f(z)) = 0 \).