Complex Analysis Math 147—Winter 2008
Review Problems for final exam; March 14,2008

1. Let f be entire and suppose that the second derivative of f is bounded: $|f''(z)| \leq M$ for all $z \in \mathbf{C}$. Prove that f is a polynomial of degree at most 2.

   Hint: Use Liouville's theorem on f'' and then the fact that in a polygonally connected open set, an analytic function whose derivative vanishes everywhere must be a constant.

2. Suppose that f is entire and that $|f(z)| \leq |z|^{2.5}$ for all $|z| > 100$.
   (a) Given z_0, show that for R sufficiently large, $|f^{(3)}(z_0)| \leq 3!R^{3.5}/(R - |z_0|)^4)$
   (b) Prove that f must be a polynomial of degree at most 2.

   Hint: Use the Cauchy integral formula for the third derivative of f and estimate the integral using the fact that since $|z| = R$, $|z - z_0| \geq R - |z_0|$

3. Let f be an entire function and let $a, b \in \mathbf{C}$.
   (a) Evaluate the integral $\int_{|z|=R} \frac{f(z)}{(z-a)(z-b)}\, dz$ if $R > |a|$ and $R > |b|$.
   (b) Use the result of (a) to give a proof of Liouville's theorem.

   Hint: Estimate the integral and let $R \to \infty$.

4. (a) Show that the following function is analytic on the open unit disk: $f(z) = \int_0^1 \frac{dt}{1-tz}$

   Hint: Use Morera's theorem and an interchange of the order of integration.

   (b) Find a power series expansion for this function.

   Hint: Use the known power series for the integrand and interchange the summation and integration.

5. Let $0 < r < R$ and $A := \{z \in \mathbf{C} : r \leq |z| \leq R\}$. Show that there is a positive number ϵ such that for each polynomial p.

$$\sup\{|p(z) - z^{-1}| : z \in A\} \geq \epsilon.$$

   Hint: Supposing it is not true, either find a sequence of polynomials converging uniformly to $1/z$ on compact subsets of A, then integrate over a circle, or alternatively, use the estimate on contour integrals.

6. Let f_n be a sequence of functions which are continuous on the closed unit disk $\{|z| \leq 1\}$ and analytic on the open disk $\{|z| < 1\}$. Suppose the f_n converges uniformly to a function f on the unit circle $\{|z| = 1\}$. Show that f can be extended to a function g on $\{|z| \leq 1\}$ which is analytic on $\{|z| < 1\}$.

   Hint: Apply the maximum modulus theorem to $f_n - f_m$.

7. (a) Prove that if f is an automorphism of the open unit disk (that is, $f : \{|z| < 1\} \to \{|z| < 1\}$ is analytic, one-to-one and onto) and if $f(0) = 0$, then $f(z) = e^{i\theta}z$ for all $|z| < 1$

    Hint: Assuming, as you may, that f^{-1} is analytic, use Schwarz's lemma for f and f^{-1}.

   For (b) and (c), let D be an open subset of $\mathbf{C}$ and fix a point $a \in D$.

   (b) Show that there is at most one analytic function $f : D \to \{|z| < 1\}$ which is one-to-one and onto and satisfies $f(a) = 0$ and $f'(a) > 0$.

    Hint: If f and g are two such functions, then $f \circ g^{-1}$ is an automorphism of the open unit disk. (Also, recall that $(g^{-1})'(g(z)) = 1/g'(z)$).

   (c) Let g be any one-to-one analytic function mapping D onto the open unit disk. Show that

$$g(z) = \frac{e^{i\theta}f(z) + \alpha}{1 + \overline{\alpha}e^{i\theta}f(z)} \qquad (z \in D)$$

   where α and θ are defined by $\alpha = g(a)$ and $g'(a) = e^{i\theta}|g'(a)|$.

    Hint: Let $h := \varphi_{-\alpha} \circ g$ and evaluate $h'(a)$, where as usual $\varphi_\alpha(z) = \frac{z+\alpha}{1+\overline{\alpha}z}$

8. Let f be analytic on an open polygonally connected set D containing an open interval I of the real axis. Suppose that D is symmetric about the real axis: $z \in D \Leftrightarrow \overline{z} \in D$, and that f is real-valued on I. Prove that $\overline{f(z)} = f(\overline{z})$ for all $z \in D$.

    Hint: Define $g(z) = \overline{f(\overline{z})}$. Show, by using the Cauchy-Riemann equations that g is analytic on D. Then use the identity theorem to show that $f = g$.

9. Show that there does not exist a one-to-one analytic function of $\{z \in \mathbf{C} : 0 < |z| < 1\}$ onto the open unit disk $\{z \in \mathbf{C} : |z| < 1\}$.

    Hint: If f is such a function, show first that it would have an analytic extension g to $z = 0$. Then consider the cases $|g(0)| = 1$ and $|g(0)| < 1$. In the latter case there exists β with $0 < |\beta| < 1$ and $f(\beta) = g(0)$. (Note that $f(\beta) = g(\beta)$.) Obtain a contradiction by justifying the following steps:
    - For $\epsilon > 0$, there exists $\delta > 0$ such that $f^{-1}(B(g(0), \delta)) \subset B(\beta, \epsilon)$
    - For this δ, there exists $\delta' > 0$ such that $g(B(0, \delta')) \subset B(g(\beta), \delta)$ and hence $f(B(0, \delta') - \{0\}) \subset B(g(\beta), \delta)$
    - $B(0, \delta') - \{0\} \subset B(\beta, \epsilon)$

10. Suppose f is analytic and zero-free in a simply connected domain D. Show there is an analytic function g in D with $e^{g(z)} = f(z)$.

    Hint: Fix $z_0 \in D$ and define $h(z) = \int_{z_0}^{z} \frac{f'(w)}{f(w)}\,dw$ where the integral is taken over any curve in D from z_0 to z. Explain why h is well-defined and analytic in D with derivative $h' = f'/f$ and then show that $\frac{d}{dz}(e^{-h(z)}f(z)) = 0$.