Problem 1 Can \( f(x, y) = \frac{\sin(x+y)}{x+y} \) (for \( y \neq -x \)) be made continuous by suitably defining it on the line \( y = -x \)?

Solution: YES. Define \( g(x, y) = f(x, y) \) if \( x + y \neq 0 \) and \( g(x, y) = 1 \) if \( x + y = 0 \). \( g \) is obviously continuous for every \((x, y)\) with \( x + y \neq 0 \). You need to show that \( g \) is continuous at points \((x_0, y_0)\) with \( x_0 + y_0 = 0 \). Let \((x, y) \to (x_0, y_0)\). Then \( x + y \to x_0 + y_0 = 0 \). Moreover, \( 1 - g(x, y) = 0 \) if \( x + y = 0 \) and \( 1 - g(x, y) = 1 - \frac{\sin(x+y)}{x+y} \) if \( x + y \neq 0 \). Hence

\[
\lim_{(x,y) \to (x_0,y_0)} (1 - g(x,y)) = 0.
\]

Problem 2 Suppose that \( q \in \mathbb{R}^n \) and \( r > 0 \). Show that the function \( f \) defined by \( f(p) = |p - q|/r \) if \( |p - q| \leq r \) and \( f(p) = 1 \) if \( |p - q| > r \) is continuous on \( \mathbb{R}^n \). Hint: Show that if \( p_k \to p_0 \), then \( f(p_k) \to f(p_0) \) by considering three cases: \( |p_0 - q| < r \); \( |p_0 - q| > r \); \( |p_0 - q| = r \).

Solution: We show that if \( p_k \to p_0 \), then \( f(p_k) \to f(p_0) \).

case 1: \( |p_0 - q| < r \): Since \( p_k \to p_0 \), \( \exists N \) such that \( |p_k - p_0| < r \) for \( k > N \). For such \( k \), \( f(p_k) = |p_k - q|/r \to |p_0 - q|/r = f(p_0) \) by continuity of \(|\cdot|\).

case 2: \( |p_0 - q| > r \): Since \( p_k \to p_0 \), \( \exists N \) such that \( |p_k - p_0| > r \) for \( k > N \). For such \( k \), \( f(p_k) = 1 \to 1 = f(p_0) \).

case 3: \( |p_0 - q| = r \): \( f(p_0) - f(p_k) = 1 - f(p_k) = 0 \) if \( |p_k - q| > r \) and \( f(p_0) - f(p_k) = 1 - |p_k - q|/r \) if \( |p_k - q| \leq r \). Thus

\[
\lim_{k \to \infty} (f(p_0) - f(p_k)) = 0.
\]

Problem 3 Is the function \( f(x, y) = x/y + y/x \) differentiable at each point in its domain? Is it of class \( C^1 \) there?

Solution: The domain of \( f \) is \( \{(x, y) \in \mathbb{R}^2 : xy \neq 0\} \). Writing \( z = \frac{x}{y} + \frac{y}{x} = \frac{x^2 + y^2}{xy} \), one calculates the partial derivatives and sees that they are both continuous on this domain:

\[
\frac{\partial f}{\partial x} = \frac{x^2 - y^2}{x^2y}, \quad \frac{\partial f}{\partial y} = \frac{y^2 - x^2}{xy^2}.
\]

Hence \( f \) is of class \( C^1 \) and therefore differentiable.

Problem 4 Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) and \( T : \mathbb{R}^n \to \mathbb{R}^m \) are differentiable. Show that the transformation \( S(x) = f(x)T(x) \) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is differentiable and that for each vector \( x_0 \in \mathbb{R}^n \),

\[
S'(x_0) = f(x_0)T'(x_0) + [T(x_0)]^t f'(x_0).
\]

Solution: Let the coordinate functions of \( T \) be \( g_1, \ldots, g_m \), that is \( T = (g_1, \ldots, g_m) \). Then the coordinate functions of \( S \) are \( f_1, \ldots, f_m; \ S = (f_1, \ldots, f_m) \). Now \( D_j(f_i)(x_0) = f(x_0)D_jg_i(x_0) + D_jf(x_0)g_i(x_0) \), so that \( S'(x_0) \) is the \( m \times n \) matrix \( [D_j(f_i)(x_0)] = [f(x_0)D_jg_i(x_0)] + [D_jf(x_0)g_i(x_0)] = f(x_0)[D_jg_i(x_0)] + [T(x_0)]^t f'(x_0) \). (Write it out!)

Problem 5 Let \( S(u, v, w) = (e^{u-v}, \cos(v + u) + \sin(u + v + w)) \) and \( T(x, y) = (e^{x}, \cos(y - x), e^{-y}) \). Calculate \( (S \circ T)'(0,0) \).

Solution:

\[
S \circ T(x, y) = [e^{x-e^{y-x}}, \cos(e^{y-x} + e^{x}) + \sin(e^{x} + \cos(y - x) + e^{-y})] = [e^{x-e^{y-x}}, \cos(e^{y-x} + e^{x}) + \sin(e^{x} + \cos(y - x) + e^{-y})]
\]
\[ T'(x, y) = \begin{bmatrix} e^x & 0 \\ \sin(y - x) & -e^{-y} \\ 0 & -
\end{bmatrix} \]

\[ T(0, 0) = (1, 1, 1), \quad T'(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \]

\[ S'(u, v, w) = \begin{bmatrix} e^{x-v} & -e^{x-v} & 0 \\ -\sin(v + u) + \cos(u + v + w) & -\sin(v + u) + \cos(u + v + w) & \cos(u + v + w) \\ 0 & -1 & 0 \end{bmatrix} \]

\[ S'(1, 1, 1) = \begin{bmatrix} 1 & -1 & 0 \\ -\sin 2 + \cos 3 & -\sin 2 + \cos 3 & \cos 3 \\ 0 & 0 & -1 \end{bmatrix} \]

\[ (S \circ T)'(0, 0) = S'(T(0, 0)) \times T'(0, 0) = S'((1, 1, 1)) \times T'(0, 0) \]

Problem 6 Is it possible to solve the system of equations

\[
\begin{align*}
xy^2 + xzu + yv^2 &= 3 \\
u^2yz + 2xv - u^2v^2 &= 2
\end{align*}
\]

for \(u\) and \(v\) in terms of \((x, y, z)\) near \((x, y, z) = (1, 1, 1)\). If possible, compute \(\frac{\partial v}{\partial y}\) at \((x, y, z) = (1, 1, 1)\).

Solution: YES! Set \(F(x, y, z, u, v) = xy^2 + xzu + yv^2 - 3\) and \(G(x, y, z, u, v) = u^2yz + 2xv - u^2v^2 - 2\) and note that \(F(1, 1, 1, 1, 1) = 0\). Also

\[ \frac{\partial(F, G)}{\partial(u, v)} = \det \begin{bmatrix} xz & 2uv \\ 3u^2yz - 2uv^2 & 2x - 2uv^2 \end{bmatrix} \]

so

\[ \frac{\partial(F, G)}{\partial(u, v)}(1, 1, 1, 1, 1) = -2 \]

From \(F(x, y, z, u, v) = 0\), by the chain rule,

\[ \frac{\partial F}{\partial x} \cdot 0 + \frac{\partial F}{\partial y} \cdot 1 + \frac{\partial F}{\partial z} \cdot 0 + \frac{\partial F}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial F}{\partial v} \frac{\partial v}{\partial y} = 0 \] (1)

From \(G(x, y, z, u, v) = 0\), by the chain rule,

\[ \frac{\partial G}{\partial x} \cdot 0 + \frac{\partial G}{\partial y} \cdot 1 + \frac{\partial G}{\partial z} \cdot 0 + \frac{\partial G}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial G}{\partial v} \frac{\partial v}{\partial y} = 0 \] (2)

Solving the linear system (1),(2) by Cramer’s rule, you get

\[ \frac{\partial v}{\partial y} = \frac{\det \begin{bmatrix} xz & 2uv \\ 3u^2yz - 2uv^2 & 2x - 2uv^2 \end{bmatrix}}{\det \frac{\partial(F, G)}{\partial(u, v)}} \]

and so

\[ \frac{\partial v}{\partial y}(1, 1, 1) = \frac{\det \begin{bmatrix} xz & 2uv \\ 3u^2yz - 2uv^2 & 2x - 2uv^2 \end{bmatrix}}{\det \frac{\partial(F, G)}{\partial(u, v)}}(1, 1, 1) = \frac{2}{-2} = -1 \]
Problem 7 Let \( T(x, y) = ((x^2 - y^2)/(x^2 + y^2), xy/(x^2 + y^2)) \). Does this transformation of \( \mathbb{R}^2 - \{(0, 0)\} \) to \( \mathbb{R}^2 \) have a local inverse near \((x, y) = (0, 1)\)?

Solution:

\[
T'(0, 1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]

and so \( \det T'(0, 1) = 0 \)

so the local invertibility theorem doesn’t give any information. However, since \( T(0, y) = (1, 0) \) for every \( y \neq 0 \), \( T \) cannot have a local inverse at \((0, 1)\).

Problem 8 Do there exist functions \( f(x, y), g(x, y) \), continuous in a neighborhood of \((0, 1)\), such that \( f(0, 1) = 1, g(0, 1) = -1 \), and such that

\[
[f(x, y)]^3 + xg(x, y) - y = 0
\]

and

\[
[g(x, y)]^3 + yf(x, y) - x = 0
\]

Solution: Set \( u = f(x, y), v = g(x, y) \), \( F(x, y, u, v) = u^3 + xv - y, G(x, y, u, v) = v^3 + yu - x. \)

Note that \( F(0, 1, 1, -1) = G(0, 1, 1, -1) = 0 \) and

\[
\frac{\partial(F, G)}{\partial(u, v)}(0, 1, 1, -1) = 9
\]

So the answer is: (ta da!) YES!.

Problem 9 If \( S \) is a compact set, prove that it contains two points \( p_0, q_0 \) such that

\[
|p_0 - q_0| = \sup\{|p - q| : p, q \in S\}.
\]

Solution: Let \( \alpha = \sup\{|p - q| : p, q \in S\} \). By definition of supremum, there are sequences \( p_k \) and \( q_k \) in \( S \) such that \( \alpha = \lim k \rightarrow \infty |p_k - q_k| \). Since \( S \) is compact, the sequence \( p_k \) has a subsequence \( p_{k_j} \) which converges to a point \( p_0 \in S \). Again, since \( S \) is compact, the sequence \( q_k \) has a subsequence \( q_{k_{j_i}} \) which converges to a point \( q_0 \in S \). We now have \( \alpha = \lim k \rightarrow \infty |p_k - q_k| = \lim i \rightarrow \infty |p_{k_{j_i}} - q_{k_{j_i}}| = |p_0 - q_0| \).

Problem 10 (a) Show that a convex set is connected.

Solution: A set is connected if every pair of its points lies in a connected subset of the set. If \( p, q \in S \), and \( S \) is convex, then by definition of convex, the line segment \( L[p, q] \subset S \). This line segment is a connected set, being a continuous image of the unit interval \([0, 1]\).

(b) Show that a polygon connected set is connected. Hint: You could use the fact that a straight line segment is a connected set.

Solution: A set is connected if every pair of its points lies in a connected subset of the set. If \( p, q \in S \), and \( S \) is polygon connected, let \( L_1 = [p, p_1], L_2 = [p_1, p_2], \ldots, L_k = [p_{k-1}, q] \) be a sequence of line segments starting at \( p \) and ending at \( q \) and lying entirely in \( S \). Since \( L_1 \cap L_2 \neq \emptyset \), \( L_1 \cup L_2 \) is connected, and by induction \( L_1 \cup \cdots \cup L_k \) is connected and contains both \( p \) and \( q \).

(c) Show that \( \mathbb{R}^2 - Q \times Q \) is connected. (\( Q = \) the rational numbers.)

Solution: We show that \( \mathbb{R}^2 - Q \times Q \) is polygon connected. Let \( p = (a, b) \in \mathbb{R}^2 - Q \times Q \) and \( q = (c, d) \in \mathbb{R}^2 - Q \times Q \).

Case 1: \( a \) and \( d \) are irrational. In this case, the polygonal path \( L[(a, b), (a, d)] \cup L((a, d), (c, d)) \) lies entirely in \( \mathbb{R}^2 - Q \times Q \).

Case 2: \( a \) is irrational and \( d \) is rational. In this case, \( c \) must be irrational and the polygonal path \( L[(a, b), (a, \pi)] \cup L((a, \pi), (c, \pi)] \cup L[(c, \pi), (c, d)] \) lies entirely in \( \mathbb{R}^2 - Q \times Q \).

Case 3: \( a \) is rational. In this case, \( b \) must be irrational and at least one of \( c, d \) is irrational. If \( c \) is irrational, then the polygonal path \( L[(a, b), (c, b)] \cup L[(c, b), (c, d)] \) lies entirely in \( \mathbb{R}^2 - Q \times Q \). If \( d \) is irrational then the polygonal path \( L[(a, b), (\pi, b)] \cup L((\pi, b), (\pi, d)] \cup L[(\pi, d), (c, d)] \) lies entirely in \( \mathbb{R}^2 - Q \times Q \).