AUTOMATIC CONTINUITY OF DERIVATIONS OF OPERATOR ALGEBRAS

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1. *Introduction*

It was conjectured by Kaplansky [**17**], and proved by Sakai [**19**], that a derivation δ of a C*-algebra $\mathfrak{A}$ is automatically norm continuous. From this, Kadison [**16**: Lemma 3] deduced that δ is continuous also in the ultraweak topology, when $\mathfrak{A}$ is represented as an algebra of operators acting on a Hilbert space. Subsequently, Johnson and Sinclair [**11**] proved the automatic norm continuity of derivations of a semi-simple Banach algebra. For earlier related results, we refer to [**20, 3, 5, 9**].

In this paper, we generalize the above results of Sakai and Kadison, by considering derivations from a C*-algebra $\mathfrak{A}$ into a Banach $\mathfrak{A}$-module $\mathscr{M}$ (definitions are given in the next section). The questions we answer arise naturally from recent work on the cohomology of operator algebras [**13, 14, 15**]. Although we express our results in cohomological notation, the present paper does not assume any knowledge of the articles just cited. It turns out that the optimal situation obtains: our derivations are automatically norm continuous (Theorem 2) and, for an appropriate class of dual $\mathfrak{A}$-modules $\mathscr{M}$, they are continuous also relative to the ultraweak topology on $\mathfrak{A}$ and the weak $*$ topology on $\mathscr{M}$ (Theorem 4). Our proof of norm continuity is a development of an argument used by Johnson and Parrott [**12**] in a particular case, and is perhaps a little simpler than the treatment used by Sakai [**19**] when $\mathscr{M} = \mathfrak{A}$.

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2. *Notation and terminology*

Except when the context indicates the contrary, it is assumed that our Banach algebras and Banach spaces have complex scalars. With $\mathscr{H}$ a Hilbert space, we denote by $\mathscr{B}(\mathscr{H})$ the $*$ algebra of all bounded linear operators acting on $\mathscr{H}$. In the context of C*-algebras, the term *homomorphism* is reserved for $*$ homomorphisms. By a *representation* of a C*-algebra $\mathfrak{A}$, we always mean a $*$ representation ϕ on a Hilbert space $\mathscr{H}$; and we assume that the linear span of the set $\{\phi(A)\,x\colon A \in \mathfrak{A},\, x \in \mathscr{H}\}$ is everywhere dense in $\mathscr{H}$. This last condition implies that the ultraweak (equivalently, weak, or strong) closure $\phi(\mathfrak{A})^-$ of $\phi(\mathfrak{A})$ contains the identity operator I on $\mathscr{H}$. The set of all positive elements of $\mathfrak{A}$ is denoted by $\mathfrak{A}^+$.

If $\mathfrak{A}$ is a Banach algebra (with unit I), and $\mathscr{M}$ is a Banach space, we describe $\mathscr{M}$ as a *Banach* $\mathfrak{A}$-*module* if there are bounded bilinear mappings $(A, m) \to Am$, $(A, m) \to mA$: $\mathfrak{A} \times \mathscr{M} \to \mathscr{M}$ such that ($Im = mI = m$ for each m in $\mathscr{M}$) and the usual associative law holds for each type of triple product, $A_1 A_2 m$, $A_1 m A_2$, $m A_1 A_2$. By a *dual* $\mathfrak{A}$-*module* we mean a Banach $\mathfrak{A}$-module $\mathscr{M}$ with the following property: $\mathscr{M}$ is (isometrically isomorphic to) the dual space of a Banach space $\mathscr{M}_*$ and, for each A

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in $\mathfrak{A}$, the mappings $m \to Am$, $m \to mA$: $\mathcal{M} \to \mathcal{M}$ are weak $*$ continuous. In these circumstances, we refer to $\mathcal{M}_*$ as the *predual* of $\mathcal{M}$, and write $\langle m, \omega \rangle$ in place of $m(\omega)$ when $m \in \mathcal{M}$ and $\omega \in \mathcal{M}_*$. If $\mathfrak{A}$ is a C*-algebra acting on a Hilbert space $\mathcal{H}$, and $\mathcal{M}$ is a dual $\mathfrak{A}$-module, we describe $\mathcal{M}$ as a dual *normal* $\mathfrak{A}$-module if, for each m in $\mathcal{M}$, the mappings $A \to Am$, $A \to mA : \mathfrak{A} \to \mathcal{M}$ are ultraweak—weak $*$ continuous.

Banach modules and dual modules provide the natural setting in which to study norm continuous cohomology of Banach algebras, while ultraweakly continuous cohomology of operator algebras is developed in the context of dual normal modules **[10, 13, 14, 15]**. The simplest example of a Banach module for a Banach algebra $\mathfrak{A}$ is $\mathfrak{A}$ itself, with Am and mA interpeted as products in $\mathfrak{A}$. When $\mathfrak{A}$ is a C*-algebra acting on a Hilbert space $\mathcal{H}$, we obtain a dual normal $\mathfrak{A}$-module by taking any ultraweakly closed subspace $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ which contains the operator products Am, mA whenever $A \in \mathfrak{A}$, $m \in \mathcal{M}$. For the predual $\mathcal{M}_*$, we can take the Banach space of all ultraweakly continuous linear functionals on $\mathcal{M}$ [7; Théorème 1 (iii), p. 38], and the weak $*$ topology on $\mathcal{M}$ is then the ultraweak topology. Examples of interest arise with $\mathcal{M} = \mathfrak{A}^-$, or $\mathcal{M} = \mathcal{B}(\mathcal{H})$.

By a *derivation*, from a Banach algebra $\mathfrak{A}$ into a Banach $\mathfrak{A}$-module $\mathcal{M}$, we mean a linear mapping $\delta : \mathfrak{A} \to \mathcal{M}$ such that

$$\delta(AB) = A\delta(B) + \delta(A)B \quad (A, B \in \mathfrak{A}).$$

We denote by $Z^1(\mathfrak{A}, \mathcal{M})$ the set of all derivations from $\mathfrak{A}$ into $\mathcal{M}$, and write $Z_c^{\ 1}(\mathfrak{A}, \mathcal{M})$ for the set of all norm continuous derivations from $\mathfrak{A}$ into $\mathcal{M}$. When $\mathfrak{A}$ is a C*-algebra acting on a Hilbert space $\mathcal{H}$ and $\mathcal{M}$ is a dual normal $\mathfrak{A}$-module, $Z_w^{\ 1}(\mathfrak{A}, \mathcal{M})$ denotes the set of all derivations, from $\mathfrak{A}$ into $\mathcal{M}$, which are ultraweak—weak$*$ continuous. An easy application of the principle of uniform boundedness shows that

$$Z_w^{\ 1}(\mathfrak{A}, \mathcal{M}) \subseteq Z_c^{\ 1}(\mathfrak{A}, \mathcal{M}).$$

The notation just introduced is taken from the cohomology theory of Banach algebras, in which derivations are the $1-$cocycles.

When X and Y are Banach spaces in duality, we denote by $\sigma(X, Y)$ the weak topology induced on X by Y.

3. *Automatic continuity theorems*

Before proving the norm continuity of all derivations from a C*-algebra $\mathfrak{A}$ into a Banach $\mathfrak{A}$-module $\mathcal{M}$, we require the following simple lemma.

LEMMA 1. *If J is a closed two-sided ideal in a C*-algebra $\mathfrak{A}$, $A \in J$, $B \in J^+$, $||B|| \leqslant 1$ and $AA^* \leqslant B^4$, then $A = BC$ for some C in J with $||C|| \leqslant 1$.*

Proof. We may assume that $\mathfrak{A}$ has an identity I, and define C_t in J, for each positive real number t, by $C_t = (B+tI)^{-1}A$. Then

$$\begin{aligned} C_t C_t^* &= (B+tI)^{-1} AA^* (B+tI)^{-1} \\ &\leqslant (B+tI)^{-1} B^4 (B+tI)^{-1} \leqslant I; \end{aligned}$$

the last inequality results easily from consideration of the functional representation of the commutative C*-algebra generated by B. Hence

$$||C_t|| = ||C_t C_t^*||^{\frac{1}{2}} \leqslant 1.$$

Moreover,

$$C_s - C_t = (t-s)(B+sI)^{-1}(B+tI)^{-1}A,$$

and

$$(C_s - C_t)(C_s - C_t)^*$$

$$= |t-s|^2 (B+sI)^{-1}(B+tI)^{-1} AA^*(B+tI)^{-1}(B+sI)^{-1}$$

$$\leqslant |t-s|^2 (B+sI)^{-1}(B+tI)^{-1} B^4 (B+tI)^{-1}(B+sI)^{-1}$$

$$\leqslant |t-s|^2 I;$$

so $||C_s - C_t|| \leqslant |t-s|$ whenever $s, t > 0$.

From the preceding discussion it follows that C_t converges in norm, as $t \to 0$, to some C in $\mathfrak{A}$. Since

$$C_t \in J, \quad ||C_t|| \leqslant 1, \quad (B+tI)\,C_t = A,$$

we have $C \in J$, $||C|| \leqslant 1$ and $BC = A$.

THEOREM 2. *If $\mathfrak{A}$ is a C^*-algebra and $\mathscr{M}$ is a Banach $\mathfrak{A}$-module, then*

$$Z^1(\mathfrak{A}, \mathscr{M}) = Z_c^{\,1}(\mathfrak{A}, \mathscr{M}).$$

Proof. With ρ in $Z^1(\mathfrak{A}, \mathscr{M})$, let J be the set of all elements A in $\mathfrak{A}$ for which the mapping

$$T \to \rho(AT) : \mathfrak{A} \to \mathscr{M}$$

is norm continuous. It is clear that J is a right ideal in $\mathfrak{A}$: and the relation

$$\rho(BAT) = B\rho(AT) + \rho(B)\,AT \quad (A \in J : B, T \in \mathfrak{A})$$

shows that J is also a left ideal. Since

$$\rho(AT) = A\rho(T) + \rho(A)\,T \quad (A, T \in \mathfrak{A}),$$

it follows that J is the set of all A in $\mathfrak{A}$ for which the mapping

$$S_A : T \to A\rho(T) : \mathfrak{A} \to \mathscr{M}$$

is norm continuous. If $A_1, A_2, \ldots \in J$, $A \in \mathfrak{A}$ and $||A - A_n|| \to 0$, then each S_{A_n} is a bounded linear operator, and

$$S_A(T) = \lim S_{A_n}(T) \quad (T \in \mathfrak{A}).$$

By the principle of uniform boundedness, S_A is norm continuous, so $A \in J$. We have now shown that J is a closed two-sided ideal in $\mathfrak{A}$.

We assert that the restriction $\rho|J$ is norm continuous. For suppose the contrary. Then, we can choose $A_1, A_2 \ldots$ in J such that

$$\sum_{n=1}^{\infty} ||A_n||^2 \leqslant 1, \qquad ||\rho(A_n)|| \to \infty.$$

With B in J defined by $B = (\Sigma A_n A_n^*)^{\frac{1}{4}}$, it follows from Lemma 1 that $A_n = BC_n$ for some C_n in J with $||C_n|| \leqslant 1$. The conditions

$$||C_n|| \leqslant 1, \, ||\rho(BC_n)|| = ||\rho(A_n)|| \to \infty$$

show that the mapping $T \to \rho(BT)$ is unbounded—a contradiction, since $B \in J$. This proves our assertion that $\rho|J$ is bounded.

We claim next that the C*-algebra $\mathfrak{A}/J$ is finite-dimensional. For suppose the contrary, so that $\mathfrak{A}/J$ has an infinite-dimensional closed commutative $*$ subalgebra $\mathscr{A}$ [**18**]. Since the carrier space X of $\mathscr{A}$ is infinite, it results easily from the isomorphism between $\mathscr{A}$ and $C_0(X)$ that there is a positive operator H in $\mathscr{A}$ whose spectrum $\operatorname{sp}(H)$ is infinite. Hence there exist non-negative continuous functions $f_1, f_2, \ldots$, defined on the positive real axis, such that

$$f_j f_k \equiv 0 \text{ if } j \neq k, \quad f_j(H) \neq 0 \ (j = 1, 2, \ldots).$$

With p the natural mapping from $\mathfrak{A}$ onto $\mathfrak{A}/J$, there is a positive element K in $\mathfrak{A}$ such that $p(K) = H$. If $S_j = f_j(K)$ $(j = 1, 2, \ldots)$, then $S_j \in \mathfrak{A}$ and

$$p(S_j^2) = p(f_j(K))^2 = [f_j(p(K))]^2 = [f_j(H)]^2 \neq 0.$$

Thus

$$S_j \in \mathfrak{A},\ S_j^2 \notin J, \qquad S_j S_k = 0 \ (j \neq k).$$

If we now replace S_j by an appropriate scalar multiple, we may suppose also that $||S_j|| \leqslant 1$.

Since $S_j^2 \notin J$, the mapping $T \to \rho(S_j^2 T)$ is unbounded. Hence there is a T_j in $\mathfrak{A}$ such that

$$||T_j|| \leqslant 2^{-j}, \quad ||\rho(S_j^2 T_j)|| \geqslant M||\rho(S_j)|| + j,$$

where M is the bound of the bilinear mapping $(A, m) \to mA : \mathfrak{A} \times \mathscr{M} \to \mathscr{M}$. With C in $\mathfrak{A}$ defined to be $\Sigma S_j T_j$, we have $||C|| \leqslant 1$ and $S_j C = S_j^2 T_j$. Hence

$$\begin{aligned} ||S_j \rho(C)|| &= ||\rho(S_j C) - \rho(S_j) C|| \\ &\geqslant ||\rho(S_j^2 T_j)|| - M||\rho(S_j)||\, ||C|| \\ &\geqslant j; \end{aligned}$$

a contradiction, since $||S_j|| \leqslant 1$ and the mapping $T \to T\rho(C)$ is bounded. This proves our assertion that $\mathfrak{A}/J$ is finite-dimensional.

Since $\rho|J$ is norm continuous and J has finite codimension in $\mathfrak{A}$, it follows that ρ is norm continuous.

Remark 3. The above proof of Theorem 2 can be abbreviated a little if, in place of Lemma 1, we appeal to the much more sophisticated factorization theorem of Johnson [**8**] and Varopoulos [**21**: Lemma 2]. However, it is perhaps desirable to give a simpler proof, within the framework of elementary C*-algebra theory. The idea of proving and exploiting the cofiniteness of the ideal J can be traced back, through [**12, 9**], to its origins in the work of Bade and Curtis [**3**].

THEOREM 4. *If ϕ is a faithful representation of a C*-algebra $\mathfrak{A}$ and $\mathscr{M}$ is a dual normal $\phi(\mathfrak{A})^-$-module, then $Z^1(\phi(\mathfrak{A}), \mathscr{M}) = Z_w^n(\phi(\mathfrak{A}), \mathscr{M})$.*

Proof. With π the universal representation of $\mathfrak{A}$, there is an ultraweakly continuous homomorphism $\tilde{\phi}$ from $\pi(\mathfrak{A})^-$ onto $\phi(\mathfrak{A})^-$ such that $\tilde{\phi}(\pi(A)) = \phi(A)$ for each A in $\mathfrak{A}$ [**6**: 12.1.5, p. 237]. The kernel of $\tilde{\phi}$ is an ultraweakly closed two-sided ideal in $\pi(\mathfrak{A})^-$, and so has the form $\pi(\mathfrak{A})^-(I - P)$ for some projection P in the centre of $\pi(\mathfrak{A})^-$ [7; Corollaire 3, p. 42]. The restriction α of $\tilde{\phi}$ to $\pi(\mathfrak{A})^- P$ is thus an isomorphism from $\pi(\mathfrak{A})^- P$ onto $\phi(\mathfrak{A})^-$ and, as such, is both isometric and ultraweakly bicontinuous [7; Corollaire 1, p. 54]. Moreover $\alpha(\pi(A)P) = \phi(A)$ for each A in $\mathfrak{A}$: so α carries $\pi(\mathfrak{A}) P$ onto $\phi(\mathfrak{A})$.

We can define a left and a right action of $\pi(\mathfrak{A})^-$ on $\mathcal{M}$ by

$$A.m = \alpha(AP)m, \quad m.A = m\alpha(AP) \quad \big(m \in \mathcal{M}, A \in \pi(\mathfrak{A})^-\big). \tag{1}$$

In this way, $\mathcal{M}$ becomes a dual normal $\pi(\mathfrak{A})^-$-module and, since $\alpha(P)$ is the identity element of $\phi(\mathfrak{A})^-$,

$$P.m = m.P = m \qquad (m \in \mathcal{M}). \tag{2}$$

Suppose that $\delta \in Z^1(\phi(\mathfrak{A}), \mathcal{M})$. By Theorem 2, δ is norm continuous, so we can define a bounded linear mapping δ_P: $\pi(\mathfrak{A}) \to \mathcal{M}$ by

$$\delta_P(A) = \delta\big(\alpha(AP)\big) \qquad \big(A \in \pi(\mathfrak{A})\big). \tag{3}$$

From (3) and (1),

$$\begin{aligned} &\delta_P(AB) - A.\delta_P(B) - \delta_P(A).B \\ &= \delta\big(\alpha(AP)\,\alpha(BP)\big) - \alpha(AP)\,\delta\,\big(\alpha(BP)\big) - \delta\big(\alpha(AP)\big)\,\alpha(BP) \\ &= 0 \quad \big(A, B \in \pi(\mathfrak{A})\big); \end{aligned}$$

so $\delta_P \in Z_c^{\,1}\big(\pi(\mathfrak{A}), \mathcal{M}\big)$. For each ω in the predual $\mathcal{M}_*$ of $\mathcal{M}$, the linear functional $\omega \circ \delta_P$ on $\pi(\mathfrak{A})$ is norm continuous and therefore [6; 12.1.3, p. 236] ultraweakly continuous. It follows that the mapping δ_P: $\pi(\mathfrak{A}) \to \mathcal{M}$ is ultraweak—weak $*$ continuous. From the Kaplansky density theorem, and the weak $*$ completeness of closed balls in $\mathcal{M}$, it follows that δ_P extends without increase in norm to an ultraweak —weak $*$ continuous linear mapping $\bar{\delta}_P$ from $\pi(\mathfrak{A})^-$ into $\mathcal{M}$. A simple continuity argument shows that $\bar{\delta}_P$ is a derivation, so $\bar{\delta}_P \in Z_w^{\,1}\big(\pi(\mathfrak{A})^-, \mathcal{M}\big)$.

By (2),

$$\bar{\delta}_P(P) = \bar{\delta}_P(P^2) = P.\bar{\delta}_P(P) + \bar{\delta}_P(P).P = 2\bar{\delta}_P(P),$$

so $\bar{\delta}_P(P) = 0$. Thus, for each A in $\pi(\mathfrak{A})$,

$$\begin{aligned} \bar{\delta}_P(AP) &= \bar{\delta}_P(A).P + A.\bar{\delta}_P(P) \\ &= \bar{\delta}_P(A) = \delta_P(A) = \delta\big(\alpha(AP)\big). \end{aligned}$$

It follows that

$$\delta(T) = \bar{\delta}_P\big(\alpha^{-1}(T)\big) \qquad \big(T \in \phi(\mathfrak{A})\big).$$

The ultraweak continuity of α^{-1} and the ultraweak—weak $_*$ continuity of $\bar{\delta}_P$ now imply that $\delta \in Z_w^{\,1}\big(\phi(\mathfrak{A}), \mathcal{M}\big)$.

We outline a second proof of Theorem 4, which is similar in its main ideas to the one given by Kadison ([16; Lemma 3]: see also [7; Lemme 4, p. 309]) in the case $\mathcal{M} = \phi(\mathfrak{A})$. Suppose $\mathfrak{A}$ is a C*-algebra acting on a Hilbert space $\mathcal{H}$, and $\mathcal{M}$ is a dual normal $\mathfrak{A}^-$-module. For each A in $\mathfrak{A}^-$ and ω in $\mathcal{M}_*$, the linear functionals

$$m \to \langle Am, \omega\rangle, \quad m \to \langle mA, \omega\rangle$$

on $\mathcal{M}$ are weak $*$ continuous. Hence we can define elements ωA and $A\omega$ of $\mathcal{M}_*$ by

$$\langle m, \omega A\rangle = \langle Am, \omega\rangle, \langle m, A\omega\rangle = \langle mA, \omega\rangle, \tag{4}$$

for all m in $\mathcal{M}$, ω in $\mathcal{M}_*$ and A in $\mathfrak{A}^-$.

With δ in $Z^1(\mathfrak{A}, \mathcal{M})$, δ is norm continuous by Theorem 2. We have to show that, for each ω in $\mathcal{M}_*$, the linear functional $f: A \to \langle \delta(A), \omega\rangle$ on $\mathfrak{A}$ is ultraweakly continuous. With $\mathfrak{A}_1{}^+$ the set of positive operators in the unit ball of $\mathfrak{A}$, it suffices to show that the restriction $f|\mathfrak{A}_1{}^+$ is continuous at 0 in the strong topology (equivalently, the strong $*$ topology, since the operators are self-adjoint) [7; Corollaire, p. 45]. For T in $\mathfrak{A}_1{}^+$,

$$\begin{aligned} f(T) &= \langle \delta(T), \omega\rangle \\ &= \langle T^{\frac{1}{2}}\delta(T^{\frac{1}{2}}) + \delta(T^{\frac{1}{2}})\, T^{\frac{1}{2}}, \omega\rangle \\ &= \langle \delta(T^{\frac{1}{2}}), \omega T^{\frac{1}{2}} + T^{\frac{1}{2}}\omega\rangle; \end{aligned}$$

whence

$$|f(T)| \leqslant \|\delta\|\,(\|\omega T^{\frac{1}{2}}\| + \|T^{\frac{1}{2}}\omega\|). \tag{5}$$

If T converges to 0 in the strong topology, the same is true of $T^{\frac{1}{2}}$, since

$$\|T^{\frac{1}{2}}x\|^2 = \langle Tx, x\rangle \leqslant \|Tx\|\,\|x\|$$

for each x in $\mathscr{H}$. Moreover, strong continuity of $f|\mathfrak{A}_1{}^+$ at 0 follows from (5) if we prove that $\|\omega T^{\frac{1}{2}}\|$ and $\|T^{\frac{1}{2}}\omega\|$ both tend to 0. Accordingly, it suffices to prove the following result, which may be of independent interest.

LEMMA 5. *If $\mathcal{R}$ is a von Neumann algebra acting on a Hilbert space $\mathscr{H}$, $\mathcal{M}$ is a dual normal $\mathcal{R}$-module and $\omega \in \mathcal{M}_*$, then the mappings*

$$A \to \omega A,\ A \to A\omega: \mathcal{R} \to \mathcal{M}_* \tag{6}$$

defined by (4) *are continuous from the unit ball of $\mathcal{R}$ (with strong $*$ topology) into $\mathcal{M}_*$ (with norm topology).*

Proof. From (4), and the ultraweak—weak $*$ continuity of the mappings $A \to Am$, $A \to mA$ from $\mathcal{R}$ into $\mathcal{M}$, it follows that the mappings (6) are continuous from $\mathcal{R}$ (with the ultraweak topology, $\sigma(\mathcal{R}, \mathcal{R}_*)$) to $\mathcal{M}_*$ (with the weak topology $\sigma(\mathcal{M}_*, \mathcal{M})$). Accordingly, these mappings are continuous also with respect to the Mackey topologies [**4**: p. 70] of $\mathcal{R}$ (in duality with $\mathcal{R}_*$) and of $\mathcal{M}_*$ (in duality with $\mathcal{M}$). The Mackey topology of $\mathcal{M}_*$ is the norm topology, and the Mackey topology on $\mathcal{R}$ coincides, on the unit ball, with the strong $*$ topology ([**2**; Theorem 11.7; **1**; Corollary 1]).

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