The Riemann Hypothesis
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Prologue

- Mathematicians form a kind of tribe, with its own language and customs. It is given to them to see truths with a clarity that is sometimes breathtaking. *Mathematical* explorers believe that there is no limit to the new discoveries they might make. Mathematicians are human too (Linear Algebra Done Right, The Joy of Sets). This book is a portrait of a particular time in mathematics, when one of math's most important problems might be on the brink of being solved. If the Riemann hypothesis is true, it would reveal a secret about prime numbers, but has no foreseeable practical consequences that could change the world.
- Mathematicians discover the pleasure of mathematics at a young age: Brian Conrey (twin primes), Charles Ryavec (cubic equation), Alexander Ivic (Lagrange interpolation), Louis de Branges (an equation in integers), Julia Robinson (square root of 2) , Alain Connes (a personal system of mathematics).
- Clay Mathematical Institute prize.

1 Prime Time

- Whole numbers are to primes what molecules are to atoms. This is the Fundamental Theorem of Arithmetic. Atoms run out before you get to a hundred, whereas the primes go on forever (Euclid, 3rd century BC).
- Adjacent primes, twin primes, largest *known* prime (Is this worth the effort?). Is there always at least one prime between successive squares? What about gaps between primes?
- Mathematicians have always wondered how the primes are distributed among the entire sequence of whole numbers. Carl Friedrich Gauss counted all the primes up to 3 million, and "saw" that the rate of change of the number of primes in a block of 1000 was characteristic of the way logarithms behave. This became know as the "Prime Number Theorem", when it was proved in 1896.
- The Prime Number Theorem forms a crucial link between the prime numbers and the Riemann hypothesis. If the Riemann hypothesis was true it could lead to an *exact* formulation of the Prime Number Theorem, instead of one that is always off by several percent.

2 Gorgeous Stuff

- Pure math is not usually associated with big money. If people have ideas about the Riemann hypothesis they'll work on them without the inducement of a prize. Unlike an artist or a musician, a mathematician can display the quality of his mind only to other like-minded people.
- Continued fractions and the golden mean (Daniel Bump).

- The consequences are fantastic: the distribution of primes. If it is not true, then the world is a very different place.
- Recap: a formula devised by Gauss which purports to calculate the number of prime numbers up to any number x (the Prime Number Theorem) is not quite accurate. If you actually count the number of primes less than any number and compare the result you get by using the formula, there is always a difference of a few percent. The expected number is known and was proven in 1896, by the Prime Number Theorem. The value of the difference is what the Riemann hypothesis provides, if it's true.
- More precisely: What is the total number of primes less than any number n? Gauss's guess: $n/\log n$—off by several percent. Riemann's first guess: $RF(n)$—off by a fraction of 1 percent. Riemann's better guess: $RF(n)$ minus the sum of the infinite series S (Zeta function)—BULLSEYE!
- Euler discovered a relationship between the zeta function, a sum which uses all the whole numbers, and a product, which uses only the prime numbers.
- Euler's zeta function and Riemann's zeta function.

3 New Numbers for Old

- If the correct steps in reasoning lead to an unfamiliar or counterintuitive answer, mathematicians sometimes see that as the starting point for a new journey. In order to advance a subject, mathematicians have had to create concepts that might on the face of it seem to make no sense—negative numbers, calculus, complex numbers.
- The "**i**" is a useful marker which enables us to identify the part of the complex number that is on an axis at right angles to the real number axis. Once **i** was invented, there were all sorts of situations in which it really *was* the square root of -1 in quite a profound sense.
- Recap: The numbers we have all grown up with are just one of many number systems in mathematics. In fact, even within the familiar real numbers there are subset such as the integers and the fractions. But mathematicians have devised other number systems, which sometimes obey the same rules. Complex numbers are at the hear of on of these systems, and we can sometimes manipulate them in similar ways to the real numbers.
- The distinction between the Euler zeta function and the Riemann zeta function.

4 Indian Summer

- Hilbert's address at the second International Congress of Mathematicians 1900 (Paris): Unsolved problems in mathematics (10 in lecture, 23 in report). These were not mere brainteasers from the puzzle pages of newspapers. Each of them came from some key field of mathematics at the time. If they were to be solved, their solutions would advance that field in new and promising directions. Some were solved in Hilbert's lifetime (before 1943); some after Hilbert's death, including "Fermat's last theorem"; some are still unsolved, including the Riemann hypothesis.
- "Nowadays (1915) there are only three really great English mathematicians: Hardy, Littlewood, and Hardy-Littlewood." They were joined by Ramanajun, a genius (1729 is the smallest number expressible as the sum of two cubes in two different ways." All three contributed to progress but the Riemann hypothesis remained elusive.
- Just as the complexity of the Riemann hypothesis arises from a simple question about prime numbers, one of Ramanujan's contributions to higher mathematics comes from asking an even simpler question concerning "sums" of the sort that any child can do (partitions). One of the things that makes number theory so captivating to mathematicians is the hidden depths that lurk beneath a placid and sometimes obvious surface. (The formula for the partition function)

- A postcard from an atheist.

5 "Very Probably"

- Sieve methods. (Examples: squares, $a^2 + b^4$).
- Many of the top mathematicians in the (Riemann hypothesis) field were gripped at an early age by prime numbers and the Riemann hypothesis (Iwaniec, Bombieri, Jutila).
- Most (nonmathematical) people have no view on the matter (whether the Riemann hypothesis is true or false). As nonmathematical readers having come this far, you still know very little about the Riemann hypothesis. Here is a summary of what you know: There is a mathematical expression that predicts roughly how many prime numbers there are smaller than any number you care to name. You know also that this prediction, by Gauss, is not entirely accurate, and that the amount by which it is wrong is the subject of another mathematical expression, devised by the German mathematician Bernhard Riemann. With Gauss's estimate, proved by two other mathematicians in 1896, and Riemann's correction, conjectured but not yet proved by anyone, we know much more about how the prime numbers are distributed. At the heart of Riemann's correction factor, and essential to understanding how it is related to prime numbers, is Riemann's zeta function and, in particular, a series of numbers which are known as the Riemann zeros.
- Explorers, diggers, guides (three types of mathematicians, all necessary for a successful proof).
- It is known that all Riemann zeros are of the form $s + it$ where $0 < s < 1$ (critical strip). The Riemann hypothesis is the assertion: $s = 1/2$ for all the zeros (critical line).
- Now that powerful computers are widely available, you can calculate as many Riemann zeros as you wish, but from Riemann's time until the 1960s, such calculations had to be done by hand. By 1935, the first 104 zeros were found. Then with more sophisticated equipment, many hundreds and soon thousands could be calculated. Every single one of them was of the form $1/2 + bi$.
- The Riemann-Siegel formula for calculating zeros—implies that Riemann himself found some of the zeros. Riemann was however very cautious—he said that 'very probably'—'all the zeros are on the critical line.' (He did not bet his life on it being true)

6 Proofs and Refutations

- In 1959 John Nash (A Beautiful Mind) announced a proof of the Riemann hypothesis. The lecture was a florid manifestation of Nash's schizophrenia, which followed his early brilliance, and preceded his remarkable but fragile recovery.
- There is no greater indication of the difficulty and importance of Riemann's Hypothesis and his zeta function than the roll call of distinguished mathematicians who have tried and failed to prove it.
- Even Alan Turing, the British mathematician who played such an important part in the British deciphering operation during the Second World War, was seduced by the fascination of the Riemann Hypothesis. Turing had decided that the hypothesis was false and started to build a machine to calculate the zeros. The more zeros you found, the better the chance to find one off the critical line! But the machine was never finished, as more pressing matters intervened—the Second World War and the need to crack the enemy's enciphered messages. Modern computers—whose conceptual origins can be traced back to the work of Turing—have shown that many billions of zeros all lie firmly on the critical line, so Turing's machine would not have helped.
- Some other high profile embarrassments: Stieltjes 1896; Rademacher 1943; Levinson 1974. So great is the desire of mathematicians to see a proof of the Riemann Hypothesis that they can easily be fooled in their eagerness. Lower profile failures: Gavrilov (Ukrainian), Matsumoto (Japanese). In between was Louis de Branges.

7 The Bieberbach Conjecture

- The popular idea of mathematics is that it is largely concerned with calculations. In fact, "mathematics is no more the art of reckoning and computation than architecture is the art of making bricks or hewing wood, no more than painting is the art of mixing colors on a palette, no more than the science of geology is the art of breaking rocks, or the science of anatomy the art of butchering."
- Bieberbach's conjecture was an important statement about functions of complex numbers, and de Branges succeeded in proving it in 1984, sixty-eight years after it was first formulated. It was no mean mathematical feat, and even mathematicians who don't take de Branges seriously today (because of his boasts about the Riemann hypothesis) do not deny the magnitude of this achievement.
- Perhaps de Branges couldn't be blamed for not knowing that the result he needed to complete the proof of the Bieberbach Conjecture had already been proved, when one of the people who had already proved it didn't even remember himself.
- Two months later, de Branges flew to St. Petersburg (at the time called Leningrad) to present his proof at the Steklov Institute, a leading center of mathematics in the Soviet Union. It seems on the surface as though the proof of the Bieberbach Conjecture was more of a joint effort than a single mathematician's achievement. However, there's nothing unusual in a mathematician producing an initial proof that has assumptions that need to be tested. The key contribution to a major proof is the vision of the mathematician who sees the broad outline of the proof.
- The question at the beginning of the twenty-first century was this: is de Branges about to pull off the same trick with the Riemann hypothesis as he did with the Bieberbach Conjecture, or are most of the world's number theorists right in dismissing his latest efforts?

8 In Search of Zeros

- Andrew Odlyzko is one of the diggers of mathematics. Like Alan Turing, but with infinitely greater means as his disposal, Odlyzko could one day prove that the Riemann Hypothesis is wrong—but never that it is right. He is compiling an ever-growing list of the values of s that make $\zeta(s)$ equal to zero.
- The connection between the Riemann hypothesis and prime numbers is not obvious. Some descriptions are: "The zeros of the Riemann zeta function are giving the Fourier transform of the set of prime numbers." "The entire collection of Riemann zeros is like a hologram of the primes."
- There is a growing role in theoretical mathematics for computers as more than just number crunchers. In 1977, the Four-Color Theorem, which had been puzzling mathematicians for over a hundred years, was finally proved—by a computer. Now, over two decades later, the first-ever computer proof of a major theorem has entered the annals of mathematics and is generally accepted. This hasn't stopped mathematicians from looking for a simpler proof that humans can work through for themselves.
- The primary use of computers is to confirm hypotheses rather than prove them, thus giving people the confidence to pursue a proof. However large the number of confirming examples, there's no guarantee that there isn't a surprise around the corner.

9 The Princeton Tea Party

- Mathematics—and indeed science in general—is often seen as a collaborative enterprize on a global scale. When Hugh Montgomery was visiting Princeton in 1972 and was introduced to the polymath scientist and mathematician Freeman Dyson over tea, he answered perfectly truthfully when Dyson asked him conversationally what he was working on. His answer struck a chord with Dyson, who then supplied a piece of information that indirectly led to what today is seen by many as the most promising approach to proving the Riemann hypothesis.

- Apart from Einstein, who preferred to be known as a physicist, there are very few famous modern mathematicians, people whose achievements are known to the public. One of the few is Andrew Wiles, who in 1995 at the age of forty-four proved Fermat's last theorem. If the Riemann hypothesis is ever proved, it is likely to attract considerably more attention (television documentary, two popular books, a musical "Fermat's last tango"). The writers of the musical managed to capture the essence of the mathematical enterprize as well as the human drama of Wiles's struggle with Fermat's Last Theorem. It embodied as much passion, frustration, and triumph as is found in the plot of any conventional film or play.
- There is no immediate financial reward in mathematics, and science and math have been traditionally open disciplines where every new result in a field is expected to be published as soon as it is verified so that other scientists or mathematicians can benefit from the discoveries in their own research. Wiles didn't want to work this way, which caused understandable distress to mathematicians who were on a similar track but further behind. Atle Selberg has been working on it since 1940 and made a number of very important contributions to the theory of primes. But he never—in public—said that he was anywhere near a proof.
- Montgomery was intrigued by one of his own discoveries (namely, that if you assume the Riemann hypothesis to be true, then the differences between pairs of zeros obey a particular rule), so he decided to make a quick trip to Princeton to consult with Selberg. It was when Montgomery mentioned this the Dyson spotted a connection between two apparently unconnected fields of knowledge—quantum physics and number theory. It turned out that physicists looking for ways to characterize the behavior of atomic particles had come up with a formula that was very similar to Montgomery's description of the zeros of the Riemann zeta function. From that conversation has come a whole new approach to the Riemann Hypothesis, and the possibility that in some quite significant way the quantum universe behaves as if it is driven by the location of the Riemann zeta function's zeros.

10 A Driven Man

- One of the surprising things about the working life of pure mathematicians is the amount of time they can spend on a particular task. When you consider the intensity and narrowness of focus of mathematicians working on a deep problem, it is extraordinary that they can spend year after year with it as their preoccupation.
- Louis de Branges, of Bieberbach fame, claimed on several occasions to have a successful approach to the Riemann hypothesis. One of these is based on a field largely of his own devising, but he had some difficulty actually proving the relevance of this new field to the Riemann Hypothesis. After ten years, he published a proof in 1985, which turned out to be wrong. Undaunted, de Branges managed to find another way to tackle the proof, lectured on it in Paris, but when he returned to the United States he found a mistake in his proof. Although this didn't endear de Branges to his colleagues, he could not be dismissed as a crank because of his Bieberbach success.
- Mathematical history is littered withe examples of the most famous mathematicians who have nevertheless made serious mistakes.
- de Branges is a driven man. He decided at an early stage in his career that he had a route to a proof of the Riemann hypothesis, and he has never lost sight of that objective.

11 The Physics of Mathematics

- No one could have predicted that a proof (of RH) might be found outside mathematics—in physics—and that the Riemann zeros bear an uncanny resemblance to the behavior of hydrogen atoms in a very strong magnetic field. No one has ever been able to relate the prime numbers to any physical system—until now.

- Until the 20th century, our understanding of the movement of matter under the influence of forces was governed by what's called classical mechanics, based on the laws described by Isaac Newton. However, the tiniest atomic particles behaved in a nonclassical way, and the new tools of quantum mechanics were devised to describe them. More recently the even newer field of chaos theory has been developed to explain an unpredictable type of behavior that occurs in physical systems, both large and small.

- A mathematical technique called random matrices is used to handle the thousands or millions of pieces of data generated when quantum mechanics is applied to a system of particles, and it looks as though the results of doing this can produce data suspiciously similar to the Riemann zeros. This marriage of quantum mechanics and number theory is much rarer than the many stories of abstract mathematical ideas which have proved to be surprisingly useful in physics or chemistry, years—or centuries—after their discovery.

- Michael Berry's remarkable idea coming from his study of quantum chaotic systems: the Riemann zeta function behaves as if there is an underlying dynamic system controlling the position of all those zeros. Over the second half of the 20th century, scientists developed a detailed understanding of how certain collections of atomic particles behaved, using the insights of quantum mechanics. But there were some types of behavior and groupings of atoms that didn't seem to obey the rules of quantum mechanics. Chaos theory was developed to describe macroscopic systems—involving larger than subatomic particles—which behaved in a way that should have been predictable but wasn't. "A system doesn't have to be complicated for its motion to be complicated. That's what chaos is all about."

- Berry and his colleagues believe that a collection of matrices associated with the chaotic behavior of certain systems of atomic particles may have characteristics that ar similar to the collection of zeros of the Riemann zeta function. "It's almost as if the Riemann zeros themselves are like physical entities."

- For a time, it seemed that random-matrix theory was enough to describe the statistics of quantum energy levels of classically chaotic systems. After a correspondence with Odlyzko, Berry made the following adjustment: "The very strong suggestion is therefore that the Riemann zeros are eigenvalues not of a random matrix, but of a matrix corresponding to a quantum system whose classical dynamics is chaotic." But others (Sarnak) believe that random-matrix theory could well produce new information, if it's possible to describe a physical system whose energy levels are the Riemann zeta zeros.

- It would be quite a coup if the most important unsolved problem in mathematics were to be solved by a physicist rather than a mathematician, and not many mathematicians think this is likely (Alain Connes has an idea of what the dynamical system is).

12 A Laudable Aim

- The RH is a big thing, and some people will be motivated by personal ambition, and they would work quietly away and not tell anybody, as Andrew Wiles did with the Fermat Theorem.

- The American Institute of Mathematics (AIM) was founded by electronics millionaire John Fry to encourage the type of cooperation that mathematicians working on the Riemann Hypothesis sometimes shy away from.

- RH is the most basic connection between addition and multiplication that there is. That connection comes in the beautiful formula discovered by Euler, where a series of terms involving all the integers are added together and shown to equal a series of terms involving the primes that are multiplied together.

- AIM director Conrey's idea for a proof uses an unusual function called the Möbius function. It can be manipulated into an expression that is equivalent to the Riemann zeta function.

- Mertens's conjecture implies RH. However, Mertens's conjecture is false (1984).

- One of the things that made math difficult for the professional mathematician trying to break new ground is that you never know how near you are to your goal.

- When Fry set up AIM, he thought that if he spent a few hundred thousand dollars flying the best people in the field to one place, putting them together for a week and telling them to prove RH, then they would. It's an attitude similar to JFK's pledge to put men on the Moon by the end of the 1960s—which worked—or Nixon's to cure cancer by a similar onslaught of brainpower—which didn't. Fry's initiative was likewise unsuccessful.

- The theory of L-functions (the zeta function is one of them) is an attempt to place RH in a more general context, which sometimes makes its study more transparent. Everything we know for Riemann's zeta function we know for the L-functions. And just as we can't prove RH for the Riemann zeta function we can't prove the Riemann hypothesis for any of these other ones.

- "Hand-waving" doesn't just mean the expressive gestures that a mathematician makes as he or she covers two large blackboards with symbols, sketches, and graphs. More fundamentally, it describes a process that is sometimes an important stage in developing an argument in front of colleagues. If a mathematician feels fairly sure that a particular step is justified, but requires a certain amount of hackwork to establish, he or she will skip ahead to the next step, then the one after, where more interesting thinking may lie. Hand-waving will almost certainly be replaced by accurate proofs at a later stage.

13 "No simple matter"

- "I am sure that Louis de Branges's many 'wrong' proofs of RH and other conjectures are as chuck-full of brilliant ideas as is his proof of Bieberbach."

- The Gamma function was important in de Branges's Bieberbach Conjecture work, and what he learned then has led him to see it as a crucial part of proving the Riemann Hypothesis.

- The real world of mathematics is far removed from that of math professors who set their students neat problems.

- In 2001, it had become increasingly clear that no one took seriously the possibility that de Branges might prove the Riemann Hypothesis.

14 Taking a Critical Line

- The Mathematical Institute at Oberwolfach in the Black Forest of Germany. Another venue for the RH industry. One characteristic of the talks at Oberwolfach is that there is always a sense of work in progress.

- The Riemann Hypothesis is a precise statement and in one sense what it means is clear, but what it's connected with, what it implies, where it comes from, can be very unobvious. Equivalent statements—mathematical statements which, if they are true, imply the Riemann Hypothesis–can often seem to have little or no connection with the Riemann zeta function. The simplest is one involving "Farey series."

- There are statements that are equivalent to RH and statements that follow from RH. There are also several different Riemann Hypotheses. In some ways, exploring the consequences of RH is a way of testing it. People spend a lot of time deriving consequences of RH. Both types of activity (conditional and unconditional) are fruitful in trying to push the subject forward.

- It seems to be possible to go in all sorts of directions from the Riemann Hypothesis itself. Its tentacles reach into all sorts of areas of mathematics. Often, ideas of this very abstract function come down to geometry, either as analogies or sometimes as an interpretation of numbers as coordinates in space. Martin Huxley believes he's found a way of linking the values in the seta function to triangles in 'hyperbolic space.'

- The Lindelöf Hypothesis, considered infinitely easier than RH, tries to provide a kind of average description of the behavior of the Riemann zeta function as the value of s changes. If you think of the zeta function as a three-dimensional landscape, a surface that meanders over a horizontal plane, occasionally dipping to "sea level" at points where $\zeta(s)$ is zero, the Lindelöf Hypothesis tells us how "undulating" this surface is in the critical strip. To prove LH, a certain exponent must be proved to be zero. This exponent has decreased from 0.166... (a result of Hardy and Littlewood in 1915) to 0.155 in 2001, with about twenty intermediate results. Each of these tiny improvements on the third or fourth decimal place needed new ideas, fresh views, and this is perhaps the best example to show how unbelievably difficult RH is.

15 Abstract Delights

- The glorious achievements of math are less accessible than those of almost any other aspect of human culture. It is a measure of the subtlety of the issues raised by RH that a statement which starts with a relationship between the integers and the primes can end up in areas of abstraction that are every bit as rarefied as Dr. Swartz's notes (a book with an evocative title whose blurb has barely a word or expression comprehensible to someone who hasn't done a postgraduate math course).
- Math gets interesting only when it takes off from the concrete and soars into the realms of abstraction. How do mathematicians acquire the taste for such a rarefied diet, and, if society continues to need such people, how is such a taste to be created and nurtured? It certainly helps if someone with a latent talent for mathematics comes into contact with a good teacher. Another approach: the College of Creative Studies at UCSB.
- If RH is proved, it will surely be by people like some of the students in CCS—brilliant, wayward, passionate—who develop a deep knowledge of a small area of math and think about it morning, noon, and night because they find it a more satisfying activity than any of the other pleasures the world has to offer.
- At the age of 89, Littlewood was in a nursing home and a friend tried to cheer him up with a math problem. "Burkholder's weak L_1 inequality" sounds like a most unlikely pick-me-up for an elderly depressed man, but it did the trick in Littlewood's case. It seemed that mathematics did help to revive his spirits and he could leave the nursing home a few weeks later. A quote from another mathematician: "If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy."
- Mathematics, as described by Bertrand Russell, is akin to philosophy or logic, and much of modern mathematics is like this.
- Alain Connes speaks of four phases of discovery (some pleasurable, some painful)—concentration, incubation, illumination, and verification. Connes earned his degree in 1973 in another field, and unlike others for whom it's been the only topic they've really ever wanted to work on, he came to RH only in 1996. Some feel that if anyone is going to prove RH, it will be him—others feel he doesn't have the depth of ideas to turn the problem around, that he is dressing up in another language a well-known difficulty in the usual language. According to Connes, mathematics is based on a duality between geometry and algebra. The former helps to find a statement, and the latter is used to formulate it, corresponding to two hemispheres of the brain (visual and linguistic).

16 Discovered or Invented?

- Profundity often lies at the heart of mathematical humor. Lewis Carroll, the author of "Alice's adventures in Wonderland," was a mathematician.

- What is the world of mathematics? Could it contain anything the human mind chooses to devise, any oddball concept that a mathematician feels like describing? Or does it have limits, imposed by the very nature of mathematics itself? Fantasy or reality? Invented or discovered?
- The prime number theorem (proved just before 1900) showed that the distribution of prime numbers had a much more intricate structure than one had imagined before this. This is a very characteristic development—you aim at a particular problem and you discover that there is much more to it than met the eye.
- In physics or natural sciences, the topic of research is something which is given from outside: it is there, but in mathematics the principal object of study is an abstraction.
- Many, perhaps most, mathematicians are realists and believe that math is "out there" in some sense rather than "in here." Would extraterrestrials recognize the unusual nature of prime numbers and , by extension, develop number theory and perhaps even prove RH?
- Alain Connes believes that mathematics is *more* real than what we think of as solid external reality, that instead of mathematics being embedded in the physical world, the physical world is embedded in mathematics. Connes's passionate realism is born from experience of the deep "realities" of mathematics.
- For many who deal with this question, the most powerful argument for the reality of mathematics is the fact that it can lead to correct descriptions of the physical world and practical applications of science. It has to be said, however, that this is not the prime motive for most mathematicians, as suggested by non other than Euclid and Archimedes.
- Whatever mathematical topic mathematicians pursue for the sheer pleasure of it could, sooner or later, turn out to give some physical description of the world or have some practical application. It's arguable that the theory of relativity wouldn't have happened if Minkowski's geometry had not been available. At the heart of new encryption methods is the problem of factoring a large number into two prime factors.

17 "What's it all about?"

- RH matters because, if it is true, it proves that there is a rule for generating the prime numbers, the building blocks of all other numbers. Bernhard Riemann identified a mathematical function, now called the Riemann zeta function, that generated another infinite set of numbers, called the zeros of the function. If those zeros all behave in the way that Riemann believed, that will allow us to describe exactly how the prime numbers are distributed.
- Over and over again, mathematicians from an increasingly wide range of mathematical fields have devised methods which have at their heart the Riemann zeta function but approach it by very different routes. One opinion is: "all tools are ready to attack it but just a penetrating idea is missing."
- There is a theoretical possibility that RH is "undecidable," that is, according to something called Gödel's Theorem, it is possible to prove the statement that RH can never be proved definitely true or definitely false. This would be as damaging as RH being false, and no one believes that this is the case.
- While the twenty or thirty mathematicians who are capable of proving RH flounder in a miasma of frustration and uncertainty, on mathematician, Louis de Branges, has the confidence that escapes the rest of them. However, he suffers from, among others, credibility problems.
- Like most—perhaps all—of the other mathematicians working on the problem, the reason de Branges spends most of his waking hours on this idea is not for money, and only a little for fame, but mainly because he believes it is true and desperately wants to prove it. In fact, he and the others might even die happy if they know that *someone* had proved it.