Prof. Kaplansky stated a conjecture that any derivation of a C*-algebra would be automatically continuous [1]. In this note, we shall show that this conjecture is in fact true.

**Theorem.** Any derivation of a C*-algebra is automatically continuous.

**Proof.** Let \( A \) be a C*-algebra, \( \alpha \) a derivation of \( A \). It is enough to show that the derivation is continuous on the self-adjoint portion \( A_s \) of \( A \). Therefore if it is not continuous, by the closed graph theorem there is a sequence \( x_n = 0 \) in \( A_s \) such that \( x_n \to 0 \) and \( x_n' \to a + ib \), where \( a \) and \( b \) are self-adjoint. First, suppose that \( a \neq 0 \) and there exists a positive number \( \lambda(> 0) \) in the spectrum of \( a \) (otherwise consider \( \{-x_n\} \)). It is enough to assume that \( \lambda = 1 \).

Then there is a positive element \( h(\|h\| = 1) \) of \( A \) such that \( hah \geq \frac{1}{2} h^2 \).

Put \( y_n = x_n + 3 \cdot \|x_n\| \cdot I \), then \( y_n \to 0 \), \( y_n' = x_n' \) and \( (h y_n h)' = h y_n h + h y_n h' \); hence \( (h y_n h)' \to h(a + ib)h \).

Therefore
\[
\| (h y_n h) - h(a + ib)h \| < \frac{1}{8} \text{ for some } n_0 \text{ ..........(1)}.\]

On the other hand
\[
h y_n h \leq 4 \|x_n\| h^2 \text{ and } \frac{1}{2} \cdot \frac{h y_n h}{4 \|x_n\|} \leq hah \text{ .................(2)}\]

Since \( \|x_n\| \cdot I + x_n \geq 0 \), \( \frac{h y_n h}{4 \|x_n\|} \geq \frac{1}{2} h^2 \).

Hence
\[
\left\| \frac{h y_n h}{4 \|x_n\|} \right\| \geq \frac{1}{2} \|h\|^2 = \frac{1}{2} \text{ ..........(3)}\]

Let \( C \) be a C*-subalgebra of \( A \) generated by \( h y_n h \) and \( I \), then by the (3) there is a character \( \varphi \) of \( C \) such that \( \varphi \left( \frac{h y_n h}{4 \|x_n\|} \right) \geq \frac{1}{2} \).
Let \( \varphi \) be an extended state of \( \varphi \) on \( A \), and \( \mathfrak{m} = \{ x \mid \varphi(x^*x) = 0, \ x \in A \} \), then \( C \cap \mathfrak{m} \) is a maximal ideal of \( C \); it can be written \( h_{y_n}h - \varphi(h_{y_n}h) \cdot 1 = u^2 - v^2 \) with \( u, v \in C \cap \mathfrak{m} (u, v \geq 0) \); hence \( (h_{y_n}h)' = u'u + uu' - vv' - vv' \), so that by the Schwartz's inequality
\[
\varphi((h_{y_n}h)') = 0 \quad \text{............................(4)}
\]

Then by the (1) and (4)
\[
|\varphi(h(a + ib)h)| < \frac{1}{8} \quad \text{.............................(5)}
\]

On the other hand by the (2)
\[
|\varphi(h(a + ib)h)| \geq \varphi(hah)
\]
\[
= \frac{1}{2} \varphi(\frac{h_{y_n}h}{4 \|x_n\|}) \geq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}
\]

; hence \( |\varphi(h(a + ib)h)| \geq \frac{1}{4} \).

This contradicts the above inequality (5), so that \( a = 0 \).

Next suppose that \( b \neq 0 \) and there exists a positive number \( \mu(>0) \) in the spectrum of \( b \) (otherwise consider \( \{-x_n\} \)). It is enough to assume that \( \mu = 1 \). Then there is a positive element \( k (\|k\| = 1) \) of \( A \) such that \( kbbk \geq \frac{1}{2} k^3 \); moreover \( \|(k_{y_n}k)' - k(a + ib)k\| < \frac{1}{8} \) for some \( n_1 \).

Let \( C_1 \) be a \( C^* \)-subalgebra of \( A \) generated by \( k_{y_n}k \) and \( I \), then there is a character \( \varphi_1 \) of \( C_1 \) such that \( \varphi_1(\frac{k_{y_n}k}{4 \|x_n\|}) \geq \frac{1}{2} \). Let \( \varphi_1 \) be an extended state of \( \varphi_1 \) on \( A \), then \( \varphi_1((k_{y_n}k)') = 0 \); hence \( |\varphi_1(k(a + ib)k)| < \frac{1}{8} \).

On the other hand
\[
|\varphi_1(k(a + ib)k)| \geq \varphi_1(kbbk) \geq \varphi_1(\frac{1}{2} k^3)
\]
\[
\geq \frac{1}{2} \varphi_1(\frac{k_{y_n}k}{4 \|x_n\|}) \geq \frac{1}{4}
\]

; hence \( |\varphi_1(k(a + ib)k)| \geq \frac{1}{4} \).

This contradicts the above inequality; hence \( b = 0 \), so that \( a + ib = 0 \).

Now we obtain a contradiction and this completes the proof.
REFERENCES


MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY.