Derivations of W^*-algebras

By Shôichirô Sakai*

1. Introduction

Let $\mathfrak{A}$ be a C^*-algebra, and let D be a derivation of $\mathfrak{A}$, i.e., a linear mapping on $\mathfrak{A}$ to $\mathfrak{A}$ satisfying $D(xy) = D(x)y + xD(y)$. D is said to be an inner derivation if there is an element a in $\mathfrak{A}$ such that $D(x) = [a, x] = ax - xa$ for $x \in \mathfrak{A}$. Otherwise, it is said to be an outer derivation.

The question of whether or not a W^*-algebra can have an outer derivation has been open for some time (cf. [10, ch. 1, p. 60]). Kaplansky [6] proved that every derivation of a type I W^*-algebra is inner. Establishing a conjecture of Kaplansky, the author [9] proved that every derivation of a C^*-algebra is bounded. Miles [7] noted that every derivation of a C^*-algebra is induced by an operator in the weak closure of some faithful representation of the algebra (a direct sum of irreducible representations). Recently, Kadison [4] showed that every derivation of a C^*-algebra on a Hilbert space $\mathfrak{H}$ is spatial (i.e., it has the form $x \longrightarrow bx - xb$ for some bounded operator b on $\mathfrak{H}$), and every derivation of a hyper-finite factor is inner. Moreover, Kadison and Ringrose [5] show that every derivation of the W^*-algebra generated by the regular representation of a discrete group is inner.

In this paper, we shall show the following: every derivation of a W^*-algebra is inner. This is the affirmative solution to Kadison's conjecture.

2. Theorems

In this section, we shall show the following theorems.

THEOREM 1. *Every derivation of a W^*-algebra is inner.*

As a corollary of Theorem 1, we have:

THEOREM 2. *Let $\mathfrak{A}$ be a C^*-algebra on a Hilbert space $\mathfrak{H}$, D a derivation on $\mathfrak{A}$, $\overline{\mathfrak{A}}$ the weak closure of $\mathfrak{A}$ on $\mathfrak{H}$, then there is a bounded operator b belonging to $\overline{\mathfrak{A}}$ such that $D(x) = [b, x]$ for all $x \in \mathfrak{A}$, where $[b, x] = bx - xb$.*

To prove Theorem 1, we shall proceed as follows. Let M be a W^*-algebra on Hilbert space $\mathfrak{H}$, D a derivation on M, M' the commutant of M in $\mathfrak{H}$, and A a maximal abelian $*$-subalgebra of M'.

LEMMA 1 (Kadison [4]). *There is a bounded operator b belonging to a*

* This research was partially supported by ONR Nonr-551 (57).

W^*-algebra (M, A) generated by M and A such that $D(x) = [b, x] = bx - xb$ for all $x \in M$.

For the proof of Lemma 1, we refer to Kadison's paper.

Put $D^*(x) = [b^*, x]$, then $D^*(x) = -[b, x^*]^*$; hence D^* is also a derivation on M and $D = (D + D^*)/2 + i(D - D^*)/2i$. Therefore it is enough to assume that b is self-adjoint. Moreover, we can assume that $\|b\| = 1$. Let $B(\mathfrak{H})$ be the W^*-algebra of all bounded operators on $\mathfrak{H}$, S the unit sphere of $B(\mathfrak{H})$, M_s' the self-adjoint portion of M', then $b + M_s'$ is weakly closed in $B(\mathfrak{H})$, and so a set $K = (b + M_s') \cap S$ is weakly compact. It is easily seen that a self-adjoint element h with $\|h\| \leqq 1$ in $B(\mathfrak{H})$ belongs to K, if and only if $[h, x] = [b, x]$ for all $x \in M$.

Let M_s be the self-adjoint portion of M and define

$$r = \inf\{\|x - a\| \mid x \in M_s, a \in K\} .$$

LEMMA 2. *There are two elements x_0 and a_0 in $B(\mathfrak{H})$ such that $x_0 \in M_s$, $a_0 \in K$ and $r = \|x_0 - a_0\|$.*

PROOF. Let (x_n) (resp. (a_n)) be a sequence of M_s (resp. K) such that $\lim_n \|x_n - a_n\| = r$, then they are bounded; hence by the weak compactness of bounded spheres of $B(\mathfrak{H})$, there is an accumulate point x_0 (resp. a_0) of (x_n) (resp. (a_n)) such that $x_0 \in M_s$ and $a_0 \in K$; moreover

$$\begin{aligned} r \leqq \|x_0 - a_0\| &= \sup_{\xi \in \mathfrak{H},\ \|\xi\|=1} |<(x_0 - a_0)\xi, \xi>| \\ &\leqq \sup \overline{\lim}_{\xi \in \mathfrak{H},\ \|\xi\|=1, n} |<(x_n - a_n)\xi, \xi>| \\ &\leqq \lim_n \|x_n - a_n\| = r . \end{aligned}$$

This completes the proof.

Now let $\mathfrak{F}$ be a family of all elements $(c_\alpha \mid \alpha \in I)$ in $B(\mathfrak{H})$ such that $c_\alpha = x_\alpha - a_\alpha$, $x_\alpha \in M_s$, $a_\alpha \in K$, and $\|c_\alpha\| = r$. For $c_\alpha, c_\beta \in \mathfrak{F}$, we shall define a partial ordering as follows: $c_\alpha \prec c_\beta$ if $\|c_\alpha z\| \geqq \|c_\beta z\|$ for all $z \in Z_p$ (we do not require that the order be proper), where Z_p is the set of all central projections in M. Let $\mathfrak{F}_1 = (c_\gamma \mid \gamma \in I_1)$ be a linearly ordered subset of $\mathfrak{F}$, and put $F_\gamma = \{c_\delta \mid c_\delta \succ c_\gamma, \delta \in I_1\}$, then $\bigcap_{\gamma \in I_1} \bar{F}_\gamma \neq (\emptyset)$, where $\bar{F}_\gamma$ is the weak closure of F_γ, because $\bar{F}_\gamma$ is weakly compact. Take $c_1 \in \bigcap_{\gamma \in I_1} \bar{F}_\gamma$, then

$$\|c_1\| = \sup_{\|\xi\|=1} |<c_1\xi, \xi>| \leqq \sup_{c_\delta \succ c_\gamma, \|\xi\|=1} |<c_\delta\xi, \xi>| \leqq \sup_{c_\delta \succ c_\gamma} \|c_\delta\| = r .$$

Analogously, we have

$$\|c_1 z\| \leqq \sup_{c_\delta \succ c_\gamma} \|c_\delta z\| = \|c_\gamma z\|$$

for all $\gamma \in I_1$ and $z \in Z_p$.

On the other hand, $\|x_\gamma\| \leqq r + \|a_\gamma\| \leqq r + 1$; hence c_γ belongs to $(r+1)S \cap M_s - K$; c_1 belongs also to the set $(r+1)S \cap M_s - K$, because it is compact; hence c_1 belongs to $\mathfrak{F}$ and $c_\gamma \prec c_1$ for all $\gamma \in I_1$. Theorefore $\mathfrak{F}_1$ has an upper bound, and so by Zorn's lemma there is a maximal element $c_{\alpha_0} = x_{\alpha_0} - a_{\alpha_0}$ ($x_{\alpha_0} \in M_s$, $a_{\alpha_0} \in K$) in $\mathfrak{F}$.

LEMMA 3. $c_{\alpha_0} = 0$, *if* M' *is a countably decomposable type* III *algebra.*

PROOF. Suppose that $c_{\alpha_0} \neq 0$. Then, first of all, we shall show that there is a projection e' in M' such that $e' \sim 1 - e'$ in M' and $e'c_{\alpha_0}(1 - e') \neq 0$. Suppose that $e'c_{\alpha_0}(1 - e') = 0$ for all e' as above, then $e'c_{\alpha_0} = c_{\alpha_0}e'$; hence $ze'c_{\alpha_0} = c_{\alpha_0}ze'$ for all $z \in Z_p$, because c_{α_0} belongs to $(M, A) \subseteq (M, M')$, and so $p'c_{\alpha_0} = c_{\alpha_0}p'$ for all projection $p' \in M'$, because p' is equivalent to its central support $z(p')$ in M' and so p' can be written as a sum of two mutually orthogonal, equivalent projections $e'z(p')$, $f'z(p')$ such that $e' \sim 1 - e'$ and $f' \sim 1 - f'$; therefore $c_{\alpha_0} \in M'' = M$, and so $c_{\alpha_0} = 0$, a contradiction.

Take a projection $e' \in M'$ such that $e' \sim (1 - e')$ in M', and $e'c_{\alpha_0}(1 - e') \neq 0$. Let v' be a partial isometry of M' such that $v'^*v' = e'$ and $v'v'^* = 1 - e'$, then $B(\mathfrak{H})$ can be considered the matrix algebra of all 2×2 matrices over a W^*-algebra $\mathfrak{A} = \{e'xe' + v'e'xe'v'^* \mid x \in B(\mathfrak{H})\}$ and M' is considered a $*$-subalgebra of $B(\mathfrak{H})$ consisting of all 2×2 matrices over a W^*-algebra

$$\mathfrak{B} = \{e'xe' + v'e'xe'v'^* \mid x \in M'\}$$

(cf. [1]). Then,

$$e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad 1 - e' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let

$$c_{\alpha_0} = \begin{pmatrix} c_{11} & c_{12} \\ c_{12}^* & c_{22} \end{pmatrix},$$

then

$$e'c_{\alpha_0}(1 - e') = \begin{pmatrix} 0 & c_{12} \\ 0 & 0 \end{pmatrix} \neq 0 .$$

For $h \in K$ and $x, y \in M_s$, we have

$$\begin{aligned}
[e'(x - h)(1 - e'), y] &= - [e'h(1 - e'), y] \\
&= - e'h(1 - e')y + ye'h(1 - e') \\
&= - e'[h, y] + e'[h, y]e' \\
&= - e'[b, y] + e'[b, y]e' \\
&= 0 ;
\end{aligned}$$

therefore, $e'(x - h)(1 - e') \in M'$. Let

$$x - h = \begin{pmatrix} d_{11} & d_{12} \\ d_{12}^* & d_{22} \end{pmatrix},$$

then by the above considerations, $d_{12} \in \mathfrak{B}$.

Now let K_1 be a weakly closed convex subset of K generated by

$$\{u'^* a_{\alpha_0} u' \mid u' \in M'_u\},$$

where M'_u is the set of all unitary elements of M', then $x_{\alpha_0} - K_1$ is a weakly compact convex set and clearly $x_{\alpha_0} - K_1 \subset \mathfrak{F}$. Therefore, a subset

$$H = \left\{ d_{12} \mid x_{\alpha_0} - k = \begin{pmatrix} d_{11} & d_{12} \\ d_{12}^* & d_{22} \end{pmatrix}, \quad k \in K_1 \right\}$$

of $\mathfrak{B}$ is also weakly compact in the W^*-algebra $\mathfrak{B}$.

Now let u, v be two unitary elements of $\mathfrak{B}$, then $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ is a unitary element w' of M' and

$$\begin{pmatrix} u^* & 0 \\ 0 & v^* \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{12}^* & c_{22} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u^* c_{11} u & u^* c_{12} v \\ v^* c_{12}^* u & v^* c_{22} v \end{pmatrix};$$

therefore H contains all elements $u^* c_{12} v$ for $u, v \in \mathfrak{B}_u$, where $\mathfrak{B}_u$ is the set of all unitary elements of $\mathfrak{B}$. Hence H contains $T c_{12} T$, where T is the unit sphere of $\mathfrak{B}$, because $\mathfrak{B}$ is $*$-isomorphic to the W^*-algebra $e'M'e'$, and so $\mathfrak{B}_u$ is weakly dense in T(3). Therefore $c_{12} c_{12}^* \in H$, because $\| c_{12} \| \leqq \| c_{\alpha_0} \| \leqq 1$. Take a non-zero projection p in $\mathfrak{B}$ such that $c_{12} c_{12}^* \geqq \lambda p$ for some positive number λ, and let $z(p)$ be the central support of p in $\mathfrak{B}$, then $p \sim z(p)$ in $\mathfrak{B}$. Let v be a partial isometry of $\mathfrak{B}$ such that $v^* v = z(p)$ and $vv^* = p$, then $v^* c_{12} c_{12}^* v \geqq \lambda v^* p v = \lambda z(p)$; clearly $v^* c_{12} c_{12}^* v \in H$. Take $c_{\alpha_1} \in x_{\alpha_0} - K_1$ such that

$$c_{\alpha_1} = \begin{pmatrix} d_{11} & v^* c_{12} c_{12}^* v \\ v^* c_{12} c_{12}^* v & d_{22} \end{pmatrix}.$$

Since $x_{\alpha_0} - u'^* a_{\alpha_0} u' = u'^* (x_{\alpha_0} - a_{\alpha_0}) u' = u'^* c_{\alpha_0} u'$ for all $u' \in M'_u$,

$$\| (x_{\alpha_0} - u'^* a_{\alpha_0} u') z \| = \| c_{\alpha_0} z \|$$

for all $z \in Z_p$, and so for $k \in K_1$,

$$\| (x_{\alpha_0} - k) z \| \leqq \| c_{\alpha_0} z \|$$

for all $z \in Z_p$, this implies $c_{\alpha_0} \prec x_{\alpha_0} - k$ for all $k \in K_1$.

On the other hand, let z_1 be a central projection of M' such that

$$z_1 = \begin{pmatrix} z(p) & 0 \\ 0 & z(p) \end{pmatrix},$$

then

$$\begin{aligned}
\| c_{\alpha_1} z_1 \| &= \left\| \begin{pmatrix} z(p) & 0 \\ 0 & z(p) \end{pmatrix} \begin{pmatrix} d_{11} & v^* c_{12} c_{12}^* v \\ v^* c_{12} c_{12}^* v & d_{22} \end{pmatrix} \right\| \\
&\geqq \left\| \begin{pmatrix} z(p) & 0 \\ 0 & z(p) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d_{11} & v^* c_{12} c_{12}^* v \\ v^* c_{12} c_{12}^* v & d_{22} \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} z(p) d_{11} & z(p) v^* c_{12} c_{12}^* v \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} z(p) d_{11} & z(p) v^* c_{12} c_{12}^* v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} d_{11} z(p) & 0 \\ z(p) v^* c_{12} c_{12}^* v & 0 \end{pmatrix} \right\|^{1/2} \\
&= \| z(p) d_{11}^2 z(p) + z(p)(v^* c_{12} c_{12}^* v)^2 \|^{1/2} \\
&\geqq (\| z(p) d_{11} \|^2 + \lambda^2)^{1/2} > \| z(p) d_{11} \| \,.
\end{aligned}$$

Analogously,

$$\begin{aligned}
\| c_{\alpha_1} z_1 \| &\geqq \left\| \begin{pmatrix} z(p) & 0 \\ 0 & z(p) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_{11} & v^* c_{12} c_{12}^* v \\ v^* c_{12} c_{12}^* v & d_{22} \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} 0 & 0 \\ z(p) v^* c_{12} c_{12} v & z(p) d_{22} \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} 0 & 0 \\ z(p) v^* c_{12} c_{12}^* v & z(p) d_{22} \end{pmatrix} \begin{pmatrix} 0 & z(p) v^* c_{12} c_{12}^* v \\ 0 & d_{22} z(p) \end{pmatrix} \right\|^{1/2} \\
&= \| z(p) d_{22}^2 z(p) + z(p)(v^* c_{12} c_{12}^* v)^2 \|^{1/2} \\
&> \| z(p) d_{22} \| \,.
\end{aligned}$$

Moreover,

$$\begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix} = \frac{1}{2} \left\{ \begin{pmatrix} d_{11} & v^* c_{12} c_{12}^* v \\ v^* c_{12} c_{12}^* v & d_{22} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} d_{11} & v^* c_{12} c_{12}^* v \\ v^* c_{12} c_{12}^* v & d_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} .$$

Since $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a unitary element u_0' of M',

$$c_{\alpha_2} = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix} \in x_{\alpha_0} - K_1 \,.$$

The maximality of c_{α_0} and $c_{\alpha_1} \succ c_{\alpha_0}$, $c_{\alpha_2} \succ c_{\alpha_0}$ imply that c_{α_1} and c_{α_2} are also maximal; hence $\| c_{\alpha_1} z_1 \| = \| c_{\alpha_2} z_1 \|$.

On the other hand,

$$\| c_{\alpha_2} z_1 \| = \left\| \begin{pmatrix} d_{11} z(p) & 0 \\ 0 & d_{22} z(p) \end{pmatrix} \right\|$$
$$= \max \{\| z(p) d_{11} \| , \| z(p) d_{22} \|\}$$
$$< \| d_{\alpha_1} z_1 \| ,$$

a contradiction. Hence $c_{\alpha_0} = 0$, and this completes the proof.

Now we shall prove Theorem 1.

PROOF OF THEOREM 1. First of all, we shall suppose that M' is a countably decomposable type III algebra, then by Lemma 3, $c_{\alpha_0} = x_{\alpha_0} - a_{\alpha_0} = 0$; hence $[x_{\alpha_0}, x] = [a_{\alpha_0}, x] = D(x)$ for $x \in M$; therefore D is inner.

Next suppose that M is arbitrary type III algebra. Take a countably decomposable projection p' in M' and put $\bar{D}(x p') = D(x)p'$ for $x \in M$, then $\bar{D}$ is a derivation on a W^*-algebra Mp' on a Hilbert space $p'\mathfrak{H}$ and $(Mp')' = p'M'p'$; hence there is an element $y \in M$ such that $[yp', xp'] = \bar{D}(x p')$ for $x \in M$; hence $[y, xp'] = [b, x]p'$ for $x \in M$ and so $[y, x]z = [b, x]z$ for $x \in M$, where z is the central support of p'. This implies $bz = yz + m'z$ $(m' \in M')$; hence by the well known theorem of W^*-algebras, a uniformly closed convex subset C generated by $\{u'^* m' z u' \mid u' \in M'_u\}$ has a non-void intersection with the center Z of M'. Take $az \in C \cap Z\, (a \in M)$, then $yz + az \in M$ and $\| yz + az \| \leqq \| bz \| \leqq 1$ and $[yz + az, x] = [yz, x]$ for all $x \in M$. Therefore we can choose a family of orthogonal central projections $(z_\alpha \mid \alpha \in J)$ in M snch that for each $\alpha \in J$, there is an element y_α in Mz_α such that $\| y_\alpha \| \leqq 1$, $[y_\alpha, x] = D(x)z_\alpha$ for all $x \in M$ and $\sum_{\alpha \in J} z_\alpha = 1$. Take $y_0 = \sum_{\alpha \in J} y_\alpha$, then $y_0 \in M$, and

$$[y_0, x] = \textstyle\sum_{\alpha \in J} [y_\alpha, x] = \sum_{\alpha \in J} D(x) z_\alpha = D(x)$$

for $x \in M$; hence D is inner.

Finally suppose that M is arbitrary W^*-algebra. Take a type III factor N on a Hilbert space $\mathfrak{H}_1$, and consider the tensor product $M \otimes N$, then $M \otimes N$ is of type III (8).

Then $b \otimes 1_{\mathfrak{H}_1}$ is a bounded operator on $\mathfrak{H} \otimes \mathfrak{H}_1$, where $1_{\mathfrak{H}_1}$ is the identity operator on $\mathfrak{H}_1$. Consider $[b \otimes 1_{\mathfrak{H}_1}, x]$ for $x \in M \otimes N$, then it is a derivation on $M \otimes N$, because $[b \otimes 1_{\mathfrak{H}_1}, c \otimes d] \in M \otimes N$ for $c \in M$ and $d \in N$, so that there is an element h in $M \otimes N$ such that $[b \otimes 1_{\mathfrak{H}_1}, x] = [h, x]$ for all $x \in M \otimes N$. Hence $[h, 1_{\mathfrak{H}} \otimes N] = 0$, where $1_{\mathfrak{H}}$ is the identity operator on $\mathfrak{H}$; therefore $h \in (1_{\mathfrak{H}} \otimes N)' \cap (M \otimes N)$.

$$\{(1_{\mathfrak{H}} \otimes N)' \cap (M \otimes N)\}' \supset (1_{\mathfrak{H}} \otimes N,\ M' \otimes N')$$
$$= M' \otimes (N', N) = M' \otimes B(\mathfrak{H}_1) ;$$

hence $(1_{\mathfrak{H}} \otimes N)' \cap (M \otimes N) \subset (M' \otimes B(\mathfrak{H}_1))' = M \otimes 1_{\mathfrak{H}_1}$, and so $h \in M \otimes 1_{\mathfrak{H}_1}$, this implies that D is inner.

This completes the proof.

PROOF OF THEOREM 2. Let $\mathfrak{A}$ be a C^*-algebra on a Hilbert space $\mathfrak{H}$, D a derivation on $\mathfrak{A}$, then by the result of Kadison (4), D can be extended to a derivation D on the weak closure $\overline{\mathfrak{A}}$ of $\mathfrak{A}$, then there is an element $x_0 \in \overline{\mathfrak{A}}$ such that $[x_0, x] = D(x)$ for $x \in \mathfrak{A}$.

This completes the proof.

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(Received April 5, 1965)