Part 1

~~mazurulam hilbert.pdf~~ 3

$\left(f(x), f(y)\right) = (x|y)$

scan 111617 full. pdf

step 1 see p. 3'

$\cancel{f(x) - f(y)}$

step 2

$$\| f(x+y) - f(x) - f(y) \|^2 = \left(f(x+y) - f(x) - f(y), f(x+y) - f(x) - f(y) \right)$$

$$(x+y, x+y) \quad -(x, x+y) \quad -(y, x+y)$$

$$\left(f(x+y), f(x+y) \right) - \left(f(x), f(x+y) \right) - \left(f(y), f(x+y) \right) \cancel{+ (f(x), f(y))}$$

o$\circ$ f is additive

$$\cancel{-(f(x+y), f(x))}$$

$$\cancel{-(f(x), f(x+y))}$$

$$-(x+y, x) \quad + (x, x) \quad + (x, y)$$

$$- f(x+y) \cdot f(x) + f(x) f(x) + f(x) f(y)$$

$$= 0$$

$$- f(y) f(x+y) + f(y) f(x) + f(y) f(y)$$

$$-(y, x+y) \quad + (y, x) \quad + (y, y) = 0$$

step 3 $\left(f(\lambda x) - \lambda f(x), f(\lambda x) - \lambda f(x) \right)$

$$= \left(f(\lambda x), f(\lambda x) \right) - \lambda \left(f(x), f(\lambda x) \right) - \lambda f(\lambda x) f(x) + \lambda^2 f(x), f(x)$$

$$(\lambda x, \lambda x) - \lambda (x, \lambda x) - \lambda (\lambda x, x) + \lambda^2 (x, x) = 0$$

$\therefore$ f is homogeneous

step 1 ?

$$(f(x) \mid f(x)) = (x \mid x) \Rightarrow (f(x), f(y)) = (x, y)$$

$$\| f(x) + f(y) \|^2 = \| f(x) \|^2 + 2(f(x), f(y)) + \| f(y) \|^2$$

$$\parallel$$

$$(f(x) + f(y), f(x) + f(y)) = f(x)(f(x)) + 2(f(x)\, f(y)) + (f(y)\, f(y))$$

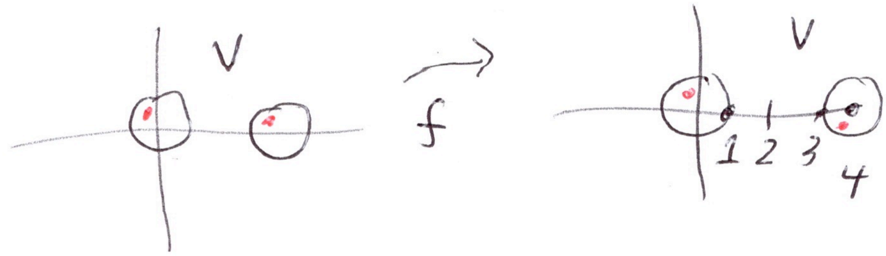
$$= (f(x) \mid f(x)) + 2(f(x)\, f(y)) + (f(y)\, f(y)) =$$

$$= \| f(x) \|^2 + 2(f(x)\, f(y)) + \| f(y) \|^2$$

$\boxed{\text{part 2}}$ 11-16-17 I (0)

part 1 11-15-17 II p. (3) - (3') mazur ulam hilbert.pdf

part 2



suppose $f = T \mid V$ is linear

$$f\left(\left(0,\tfrac{1}{3}\right) + \left(4,\tfrac{1}{3}\right) \right) = f\left(\left(4,\tfrac{2}{3}\right) \right) = \left(4,-\tfrac{2}{3}\right)$$

$$= T\left(4,\tfrac{2}{3}\right) \cancel{= T\left(0,\tfrac{1}{3}\right) + T\left(4,\tfrac{1}{3}\right)}$$

$$T\left(\left(0,\tfrac{1}{3}\right) + \left(4,\tfrac{1}{3}\right) \right) = T\left(0,\tfrac{1}{3}\right) + T\left(4,\tfrac{1}{3}\right)$$

$$\cancel{T\left(4,\tfrac{2}{3}\right)} \qquad = f\left(0,\tfrac{1}{3}\right) + f\left(4,\tfrac{1}{3}\right)$$

$$= \left(0,\tfrac{1}{3}\right) + \left(4,-\tfrac{1}{3}\right)$$

$$= (4,0) \quad \Rightarrow\Leftarrow$$

First lemma

part 3 11-16-17 p. (1) - (2)

<u>2nd lemma</u> too <u>involved</u>

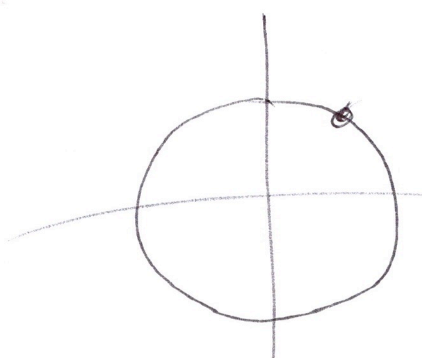
accept part 4 — to the extreme points — go to part 6 (Lemma)

part 6 lemma see 11-16-17 p. (4) - (5)

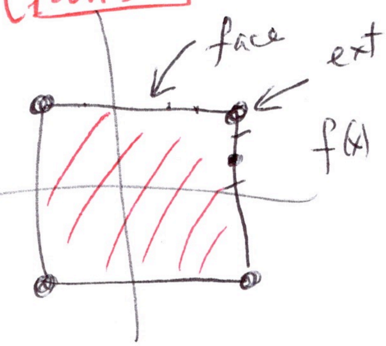
accept part 5 go to part 6 (Theorem)

this batch pp (5) - (6)

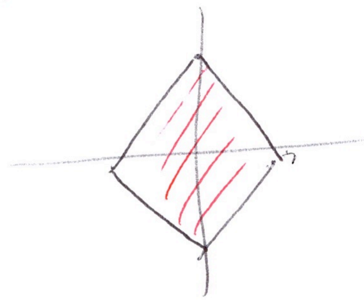
$\boxed{\text{part 3}}$



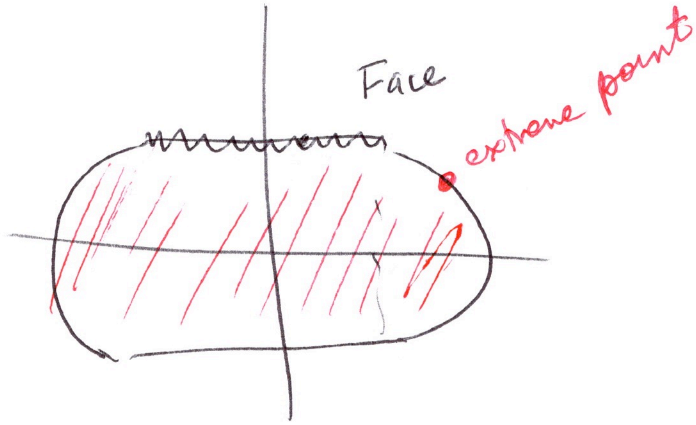
$\|x\| = (x_1^2 + x_2^2)^{1/2}$



$\|x\| = \max(x_1, x_2)$



$\|x\| = |x_1| + |x_2|$



$x \in F$

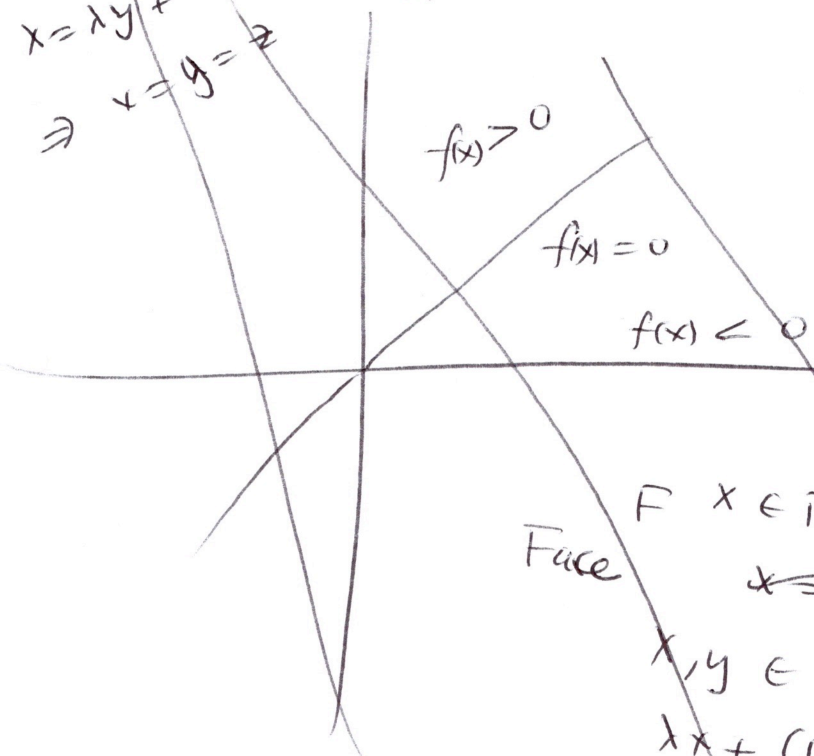
x

F

$x = \lambda y + (1-\lambda) z \Rightarrow y, z \in F$

$x = y = z$

$\Rightarrow$



$F \subset F' \subset C$

$x \in F'$

$y \in F$

$\lambda x + (1-\lambda) y \in F'$

$F \quad x \in F$

Face

$\cancel{x = \lambda y + (1-\lambda) z}$

$x, y \in C$

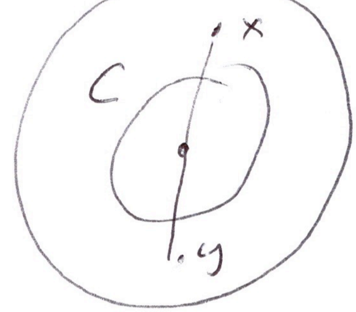
$\lambda x + (1-\lambda) y \in F$

$\Rightarrow x, y \in F$

First lemma: maximal convex set is a face

$C \subset S(X)$

S(X)



let $x, y \in S(X)$ $\quad \frac{x+y}{2} \in C$

let $C' =$ ~~the intersection of all convex~~

$co(C \cup \{x,y\}) \subset S(X)$

$= \{ \lambda a + (1-\lambda) b : \lambda \in [0,1],\ a,b \in C \cup \{x,y\} \}$

$\mu (\lambda_1 a_1 + (1-\lambda_1) b_1) + (1-\mu) (\lambda_2 a_2 + (1-\lambda_2) b_2)$

~~td~~

$co\, S = \{ \lambda x + (1-\lambda) y : x, y \in S \quad 0 \le \lambda \le 1 \}$

$x_1, x_2, y_1, y_2 \in S$

$\mu (\lambda_1 x_1 + (1-\lambda_1) y_1) + (1-\mu) (\lambda_2 x_2 + (1-\lambda_2) y_2)$

$= \underbrace{\mu \lambda_1} x_1 + \underbrace{\mu (1-\lambda_1)} y_1 + \underbrace{(1-\mu) \lambda_2} x_2 + \underbrace{(1-\mu)(1-\lambda_2)} y_2$

$\underbrace{\mu \lambda_1 + \mu(1-\lambda_1)}_{\mu} + \underbrace{(1-\mu)\lambda_2 + (1-\mu)(1-\lambda_2)}_{1-\mu} = 1$

so

$$\text{co}\, S = \{ \sum \lambda_i x_i : \sum \lambda_i = 1 \quad x_i \in S \}$$

$$\mu (\sum \lambda_i x_i) + (1-\mu) \sum_j \lambda_j' y_j$$

$$= \sum_i \mu \lambda_i x_i + \sum_j \lambda_j' (1-\mu) y_j$$

$$\sum_i \mu \lambda_i + \sum_j \lambda_j' (1-\mu)$$

$$= \mu (\sum_i \lambda_i) + (1-\mu) \sum_j \lambda_j' = 1$$

$$\sum \lambda_i x_i = \sum_{i=1}^{n-1} \lambda_i x_i + \lambda_n x_n$$

$$= \sum_{i=1}^{n-1} \lambda_i \left(\sum_{i=1}^{n-1} \frac{\lambda_i}{\sum_1^{n-1} \lambda_i} x_i \right) + \lambda_n x_n$$

---

part 6

part 6

lemma 2.2B1

$T : S(A_1) \to S(A_2)$ surjective isometry

$a, b \in S(A_1)$ $\quad ta + (1-t)b \in S(A_1)$

$$[a,b] \subset \underbrace{C_1}_{\text{maximal convex subset}} \subset S(A_1)$$

$$C_2 = \underbrace{T(C_1)}_{\text{max convex subset}} \subset S(A_2)$$

$$C_1 = v_1 + (1 - v_1 v_1^{*}) B(A_1) (1 - v_1^{*} v_1)$$

$$C_2 = v_2 + (1 - v_2 v_2^{*} B(A_2) (1 - v_2^{*} v_2)$$

$C_j = v_j + (1 - f_j) B(A_j) (1 - e_j)$ $\quad e_j\ f_j$ projections

$T_j(a) = a - v_j$ is affine isometry on A_j

$\Rightarrow$ the face C_j is affinely isometric to the unit ball $(1-f_j) B(A_j) (1-e_j)$ of the space $(1-f_j) A_j (1-e_j) \subset A_j$

$$T_0 = \quad T_1^{-1} \quad T \quad T_2$$

$$(1-f_1) B(A_1) (1-e_1) \to C_1 \longrightarrow C_2 \longrightarrow (1-f_2) B(A_2) (1-e_2)$$

By Mankiewicz T_0 is affine

and therefore so is $T|C_1 = T_2^{-1} T_0 T_1$

Thus $T \mid [a,b]$ is affine.

part 6 Theorem Let $T: S(A_1) \to S(A_2)$ be surjective isometry

Let ~~u~~ u be unitary in A_1

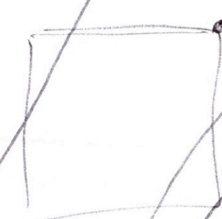
$T(u)$ has norm 1 so $T(u) = \frac{a+b}{2}$ $a, b \in S(A_2)$

~~We know that~~

is a face a maximal convex set

$F \subset S(x)$

$F \subset C \subset S(x)$



~~let x ∈~~

extreme points $(1-vv^*)B(A)(1-v^*v) = 0$

∴ extrem pt are unitary

let u be unitary in A_1 if $T(u) = \frac{a+b}{2}$ $a,b \in S(A_2)$

$\frac{\|a+b\|}{2} \le 1$

let $\|y\|=1$ $y = \frac{y+y^*}{2} + i\,\frac{y-y^*}{2}$

$z + iw$

$z^* = z$ $z - \sqrt{1-z^2}$

$1 = \|y\| \le \|z\| + \|w\|$

$$u = \frac{T^{-1}(a) + T^{-1}(b)}{2} \qquad \|T^{-1}(a)\| = \|T^{-1}(b)\| = 1$$

$$\Rightarrow u = T^{-1}(a) = T^{-1}(b) \qquad T(u) = a = b$$

$$\therefore T(U(A_1)) \subset U(A_2) \qquad T^{-1}(U(A_2)) \subset U(A_1)$$

so $\boxed{TU_1 = U_2}$

Hatori-Molnar ~~change this~~ $\exists w$ $T(u) = T(1)\,w u w^*$

or $T(1)\,w u^t w^*$

so