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Math. Annalen, Bd. 129, S. 260—264 (1955).

Derivations on Commutative Normed Algebras.

By

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A *derivation* D on an algebra is a transformation on the algebra such that

(i) $$D(a+b) = D(a) + D(b)$$

(ii) $$D(\lambda a) = \lambda D(a),\ \lambda \text{ any scalar}$$

(iii) $$D(a b) = D(a)\, b + a\, D(b).$$

We are concerned with derivations on commutative Banach algebras over the complex field, where by a Banach algebra we mean a normed algebra $\mathfrak{A}$ which is complete in its norm. The *radical* of $\mathfrak{A}$ is the intersection of all maximal ideals M in $\mathfrak{A}$ which are such that $\mathfrak{A}/M$ has a unit. If the radical reduces to the zero element, $\mathfrak{A}$ is called *semi-simple.*

A derivation on $\mathfrak{A}$ is said to be *bounded* if

(iv) $$\sup_{\|a\|=1} \|D(a)\| = \|D\| < \infty$$

Theorem 1[1]: *Let $\mathfrak{A}$ be a commutative Banach algebra and D a bounded derivation on $\mathfrak{A}$. Then D maps $\mathfrak{A}$ into its radical. In particular, if $\mathfrak{A}$ is semi-simple, $D = 0$.*

Proof of Theorem 1: A non-zero linear functional f on $\mathfrak{A}$ is called multiplicative if $f(a b) = f(a) f(b)$ for all a, b in $\mathfrak{A}$. We need the following result, due to GELFAND:

(1) If f is multiplicative, $|f(a)| \leqq \|a\|$ for each a. Since D is bounded, $\sum_{n=0}^{\infty} \frac{t^n \|D^n\|}{n!} < \infty$ if $t < \infty$, and so for any complex number λ the series $\sum_{n=0}^{\infty} \frac{\lambda^n D^n}{n!}$ converges to a bounded operator on $\mathfrak{A}$ which we call $e^{\lambda D}$. For finite-dimensional algebras it is well-known[2]) that $e^{\lambda D}(a b) = e^{\lambda D}(a)\, e^{\lambda D}(b)$ for a, b in $\mathfrak{A}$. Guided by the formal process, we proceed as follows:

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[1]) ŠILOV showed in his paper "On a property of rings of functions", Doklady Akad. Nauk SSSR. (N.S.) 58, 985—988 (1947), that the algebra of all infinitely differentiable functions on an interval cannot be normed so as to be a Banach algebra. Prof. I. KAPLANSKY conjectured that the "reason" for this was that non-zero derivations could not exist on a commutative semisimple Banach algebra. Theorem 1 proves this conjecture for bounded derivations. It seems probable that hypothesis (iv) is superfluous.

[2]) See CHEVALLEY: "Theory of Lie Groups", p. 137. Princeton Univ. Press. (1946).

Fix a multiplicative functional f and a complex number λ and set $\varphi_\lambda(a) = f(e^{\lambda D}(a))$, a in $\mathfrak{A}$. Then φ_λ is a linear functional and we claim φ_λ is multiplicative. For by (i) and (iii),

$$\frac{D^n(ab)}{n!} = \sum_{i+j=n} \frac{D^i(a)}{i!} \cdot \frac{D^j(b)}{j!}$$

Hence

$$\varphi_\lambda(ab) = \sum_{n=0}^{\infty} \frac{\lambda^n f(D^n(ab))}{n!} = \sum_{n=0}^{\infty} \lambda^n \sum_{i+j=n} \frac{f(D^i(a)) \cdot f(D^j(b))}{i! \cdot j!}$$

Also

$$\varphi_\lambda(a) \cdot \varphi_\lambda(b) = \left(\sum_{i=0}^{\infty} \frac{\lambda^i f(D^i(a))}{i!}\right)\left(\sum_{j=0}^{\infty} \frac{\lambda^j f(D^j(b))}{j!}\right)$$

Since the series in the preceding line converge absolutely, it follows that $\varphi_\lambda(a\,b) = \varphi_\lambda(a)\,\varphi_\lambda(b)$. Hence by (1), we have

(2) $|\varphi_\lambda(a)| \leqq \|a\|$ for each a in $\mathfrak{A}$ and each λ. But $\varphi_\lambda(a) = \sum_{n=0}^{\infty} \frac{\lambda^n f(D^n(a))}{n!}$ is an entire function of λ for a fixed a. By (2) this entire function is bounded in the whole plane. Hence it is a constant. Hence $f(D^n(a)) = 0$, $n \geqq 1$. In particular $f(D(a)) = 0$. But f was an arbitrary multiplicative functional and so $D(a)$ is annihilated by every multiplicative functional. Hence $D(a)$ lies in the radical, which is the assertion of the theorem.

Applications of Theorem 1:

Wielandt has shown [Über die Unbeschränktheit der Operatoren der Quantenmechanik", Math. Ann. **121**, 21 (1949—50)] that if a, b are bounded operators on a normed vector space, then $a\,b - b\,a \neq 1$. We can use Theorem 1 to strengthen this result as follows:

Corollary 1.1: Let a, b be bounded operators on a Banach space and assume that $a\,b - b\,a$ lies in the uniformly closed algebra generated by a and 1. Then $a\,b - b\,a$ is a generalized nilpotent, i. e. has a spectrum which consists only of zero.

Proof: Let $\mathfrak{A}$ be the uniformly closed algebra generated by 1 and a. For any bounded operator c, let $D(c) = c\,b - b\,c$. Then D is a derivation on the algebra of all bounded operators. We assert that D maps $\mathfrak{A}$ into itself. For $D(a) \in \mathfrak{A}$ by hypothesis. If P is a polynomial, $D(P(a)) = P'(a) \cdot D(a)$ and so $D(P(a))$ is in $\mathfrak{A}$. Finally, if c is any element of $\mathfrak{A}$, c is a limit of polynomials in a and so $D(c)$ is in $\mathfrak{A}$. Thus D is a derivation of $\mathfrak{A}$. Finally D is bounded, since $\|D(c)\| \leqq 2\,\|b\|\,\|c\|$. By Theorem 1, then, D maps $\mathfrak{A}$ into its radical. Thus $D(a)$ is in the radical of $\mathfrak{A}$. Hence by well-known results,

$$\lim_{n\to\infty} \|(D(a))^n\|^{\frac{1}{n}} = 0 \text{ and so } D(a) = a\,b - b\,a$$

is a generalized nilpotent; Q.E.D.

For other extensions of Wielandt's Theorem, see P. R. Halmos, "Commutators of operators II", Amer. J. of Math. **76**, 191—198 (1954).

Corollary 1.2[3]**:** Let C^∞ denote the algebra of all infinitely differentiable complex-valued functions on an interval. Then there exists no norm under which C^∞ is a Banach algebra.

Proof: Suppose there is such a norm. For each f in C^∞ set $Df = f'$. Then D is a derivation on C^∞. We can show that D is bounded.

For consider any point t_0 in the interval, and let φ_{t_0} be the functional which maps f into $f'(t_0)$. For $n = 1, 2, \ldots$ set $L_n(f) = \dfrac{f\left(t_0 + \frac{1}{n}\right) - f(t_0)}{\frac{1}{n}}$.

Now the maps: $f \to f(t_0)$ and $f \to f\left(t_0 + \frac{1}{n}\right)$ are multiplicative and hence bounded linear functionals. It follows that L_n is a bounded linear functional for each n. Now $\lim\limits_{n \to \infty} L_n(f) = f'(t_0) = \varphi_{t_0}(f)$. Hence by a well-known result the functional φ_{t_0} is bounded.

To show D bounded it suffices, by a theorem of BANACH, to show that if $f_n \to f$ and $Df_n \to g$ then $Df = g$. But if $f_n \to f$, $f'_n(t) \to f'(t)$ for each t by the preceding, and so

$$g(t) = \lim_{n \to \infty} f'_n(t) = f'(t) = Df(t)$$

for each t. Hence $Df = g$. Thus D is bounded.

Now C^∞ is semi-simple since $f(t) = 0$ for all t implies $f = 0$. Hence by Theorem 1, $D = 0$. But this is false. Hence the assertion of the theorem must hold.

Derivations into Larger Algebras.

Let $\mathfrak{A}$ be a commutative Banach algebra which is embedded in some larger algebra B as closed subalgebra. Let D be a (bounded) linear transformation of $\mathfrak{A}$ into B. Since $\mathfrak{A} \subseteq B$, the product of a and $D(b)$ is defined in B if a, b are in $\mathfrak{A}$. D is called a (bounded) *derivation of* $\mathfrak{A}$ *into* B if, when a_1, a_2 are in $\mathfrak{A}$,

$$D(a_1 a_2) = D(a_1)\, a_2 + a_1 D(a_2).$$

What algebras $\mathfrak{A}$ admit derivations into some commutative extension B? We need the following notion:

A (bounded) *point derivation* of $\mathfrak{A}$ is a (bounded) linear functional $d\varphi$ associated with a multiplicative linear functional φ such that

$$d_\varphi(a_1 a_2) = d_\varphi(a_1) \cdot \varphi(a_2) + \varphi(a_1)\, d_\varphi(a_2).$$

Theorem 2: *If there exists a non-zero (bounded) point derivation d_φ of $\mathfrak{A}$, then there exists a commutative extension B of $\mathfrak{A}$ and a non-zero (bounded) derivation D of $\mathfrak{A}$ into B. If $\mathfrak{A}$ is semi-simple, B can be taken to be semi-simple.*

If $\mathfrak{A} \subseteq B$ and if D is a non-zero (bounded) derivation of $\mathfrak{A}$ into B but not into the radical of B, then there exists a non-zero (bounded) point derivation of $\mathfrak{A}$.

Proof: Let B consist of all pairs (a, λ), a in $\mathfrak{A}$, λ a complex number, i.e. B is the direct sum of $\mathfrak{A}$ with the complex numbers. Multiplication is defined by:

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[3]) Originally proved by ŠILOV, cf. footnote 1).

$(a_1, \lambda_1) \cdot (a_2, \lambda_2) = (a_1 a_2, \lambda_1 \lambda_2)$. The norm in B is given by $\|(a, \lambda)\| = \max(\|a\|, |\lambda|)$. It is easy to check that B is a commutative Banach algebra and is semi-simple if $\mathfrak{A}$ is.

Let now φ be a multiplicative functional on $\mathfrak{A}$ and d an associated point derivation. Let $\mathfrak{A}_1$ be the set of all (a, λ) with $\lambda = \varphi(a)$. The map: $a \to (a, \varphi(a))$ is an algebraic isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_1$ which preserves norm since $|\varphi(a)| \leqq \|a\|$. We identify $\mathfrak{A}$ with $\mathfrak{A}_1$ so that $\mathfrak{A}$ is embedded in B, as closed subalgebra of B.

D is defined by $D\big((a, \varphi(a))\big) = (0, d(a))$. Then D is linear.

$$\begin{aligned} D\big((a_1, \varphi_1(a)) \cdot (a_2, \varphi(a_2))\big) &= D(a_1 a_2, \varphi(a_1 a_2)) \\ &= (0, d(a_1 a_2)) = (0, d(a_1)\, \varphi(a_2) + \varphi(a_1)\, d(a_2)) \\ &= (0, d(a_1))\,(a_2, \varphi(a_2)) + (a_1, \varphi(a_1))\,(0, d(a_2)). \end{aligned}$$

Hence D is a derivation. It is bounded if d is bounded.

To prove the partial converse, we note that since $D(\mathfrak{A})$ is not in the radical of B, (where D is the given derivation of $\mathfrak{A}$ into B), there exists some multiplicative functional φ on B whose restriction to $D(\mathfrak{A})$ is not zero. Define d on $\mathfrak{A}$ by $d(a) = \varphi(D(a))$. Then d is not zero and

$$\begin{aligned} d(a_1 a_2) &= \varphi(D(a_1 a_2)) = \varphi(D(a_1) \cdot a_2 + a_1 D(a_2)) \\ &= \varphi(D(a_1))\, \varphi(a_2) + \varphi(a_1)\, \varphi(D(a_2)) \\ &= d(a_1)\, \varphi(a_2) + \varphi(a_1)\, d(a_2) \end{aligned}$$

i.e. d is a point derivation on $\mathfrak{A}$. d is bounded if D is.

Note: In the construction in the preceding proof, the maximal ideal space of B was disconnected. One can however, give an example of a bounded derivation from an algebra $\mathfrak{A}$ into a larger algebra B, where the space of maximal ideals of B is connected.

Suppose $1 \in \mathfrak{A}$. Then point derivations can be interpreted in terms of ideals as follows. Let $M_\varphi = \{a \mid \varphi(a) = 0\}$, where φ is a multiplicative linear functional. Then M_φ is a maximal ideal in $\mathfrak{A}$. Let M_φ^2 be the set of linear combinations of squares of elements of M_φ and let $\overline{M_\varphi^2}$ be the closure of M_φ^2. *Then non-zero (bounded) point derivations associated with φ exist if and only if* $M_\varphi^2 \neq M_\varphi$ $(\overline{M}_\varphi^2 \neq M_\varphi)$. For if so, we can find a linear non-zero (bounded) functional l annihilating M_φ^2 and 1. Then l is a (bounded) point derivation associated with φ. For any element a in $\mathfrak{A}$ can be written as: $a = a' + \varphi(a) \cdot 1$, a' in M_φ. Then

$$\begin{aligned} l(a_1 a_2) &= l((a_1' + \varphi(a_1) \cdot 1)\,(a_2' + \varphi(a_2) \cdot 1)) \\ &= l(a_1'\, a_2' + \varphi(a_1)\, a_2' + a_1'\, \varphi(a_2) + \varphi(a_1)\, \varphi(a_2) \cdot 1) \\ &= \varphi(a_1)\, l(a_2') + l(a_1')\, \varphi(a_2). \end{aligned}$$

Conversely, if d is a non-zero (bounded) point derivation associated with the multiplicative functional φ, then $d(M_\varphi) \neq 0$ and $d(M_\varphi^2) = 0$, $(d(\overline{M}_\varphi^2) = 0)$, whence $M_\varphi^2 \neq M_\varphi$ $(\overline{M}_\varphi^2 \neq M_\varphi)$; consequently we have the:

Corollary 2.1. *Assume* $\mathfrak{A}$ *is semi-simple with unit.* $\mathfrak{A}$ *has no non-zero (bounded) derivations into a semi-simple commutative extension B if and only if* $M_\varphi^2 = M_\varphi$ $(\overline{M_\varphi^2} = M_\varphi)$ *for all multiplicative* φ.

Corollary 2.2. *The algebra* $C(X)$ *of all continuous functions on a compact Hausdorff space X has no non-zero derivations into any semi-simple commutative extension B.*

Proof: It suffices to show that $M_\varphi^2 = M_\varphi$ for all φ. Now M_φ consists of all functions f vanishing at a point x. If $f \in M_\varphi$, the real and imaginary parts of f vanish at x. Suppose f is real; then we can write $f = f^+ - f^-$ where f^+, f^- are nonnegative, continuous and vanish at x. They have continuous square roots. Hence $f \in M_\varphi^2$ and all is proved.

Added in Proof: In a paper "On the Spectra of Commutators" (Proc. Amer. Math. Soc. 5, No. 6. Dec. 1954, pp. 929—931) C. R. PUTNAM has proved the following result: "If A, B are bounded operators on a Hilbert space and $C = AB - BA$, and if $AC = CA$ and $BC = CB$, then the spectrum of C consists of 0 alone." By considering the derivation D with $D(a) = aB - Ba$ on the algebra generated by A and C, PUTNAM's theorem is readily seen to be a consequence of our Theorem 1.

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(Eingegangen am 20. Dezember 1954.)