DERIVATIONS
An introduction to non associative algebra
(or, Playing havoc with the product rule)

Series 2—Part 4
Derivations, local derivations, and 2-local derivations on
(algebras of) matrices

Colloquium
Fullerton College

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February 23, 2016
## Series 1

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Series 2

- **PART I**  JULY 24, 2014  THE REMARKABLE CONNECTION BETWEEN JORDAN ALGEBRAS AND LIE ALGEBRAS (Two theorems relating different types of derivations)

- **PART II**  NOVEMBER 18, 2014  THE REMARKABLE CONNECTION BETWEEN JORDAN ALGEBRAS AND LIE ALGEBRAS (Two theorems embedding triple systems in Lie algebras)

- (digression)  FEBRUARY 24, 2015  GENETIC ALGEBRAS

- **PART III**  JULY 15, 2015  LOCAL DERIVATIONS

- (Fall 2015 missed due to the flu)

- **PART IV** (today)  FEBRUARY 23, 2016  2-LOCAL DERIVATIONS

"Slides" for all series 1 and series 2 talks available at
http://www.math.uci.edu/INSERT a “~” HERE brusso/undergraduate.html
ABSTRACT for February 23, 2016
After reviewing derivations and local derivations we consider local and 2-local derivations on the set of 2 by 2 matrices, considered
(1) as an associative algebra under matrix multiplication;
(2) as a Lie algebra under bracket multiplication;
(3) as a Jordan algebra under circle multiplication

ABSTRACT for July 15, 2015 (for context)
A local derivation is a linear map $T$ of an algebra which at each element of the algebra agrees with some derivation $D$, which can vary from element to element. Every derivation is obviously a local derivation but not all local derivations are derivations. For a large class of operator algebras, the two notions coincide. Similar definitions and results hold for local derivations on triple systems. There is also a nonlinear version on algebras and triple systems.
AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE OVER A FIELD) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION—we are downplaying multiplication by scalars (=numbers=field elements)

ADDITION IS DENOTED BY \( a + b \) AND IS REQUIRED TO BE COMMUTATIVE \( a + b = b + a \) AND ASSOCIATIVE \( (a + b) + c = a + (b + c) \)

MULTIPLICATION IS DENOTED BY \( ab \) AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION \( (a + b)c = ac + bc, \quad a(b + c) = ab + ac \)

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (resp. COMMUTATIVE) IF THE MULTIPLICATION IS ASSOCIATIVE (resp. COMMUTATIVE) (RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)
**Table 1**  (FASHIONABLE) ALGEBRAS

**commutative algebras**  \( ab = ba \)
(Real numbers, Complex numbers, Continuous functions)

**associative algebras**  \( a(bc) = (ab)c \)
(Matrix multiplication)

**Lie algebras**  \( a^2 = 0 \), \( (ab)c + (bc)a + (ca)b = 0 \)
( Bracket multiplication on associative algebras: \([x,y] = xy - yx\) )

**Jordan algebras**  \( ab = ba \), \( a(a^2b) = a^2(ab) \)
(Circle multiplication on associative algebras: \( x \circ y = (xy + yx)/2 \) )
Types of derivations

### Derivation

\[
\delta(ab) = a\delta(b) + \delta(a)b
\]

### Lie derivation

\[
\delta([a, b]) = [a, \delta(b)] + [\delta(a), b]
\]

### Jordan derivation

\[
\delta(a \circ b) = a \circ \delta(b) + \delta(a) \circ b
\]

There are two theorems relating different types of derivations. We will use one of them extensively today (Theorem A on the next page). We gave the proofs of Theorems A and B in Series 2, part I (July 24, 2014). The proofs are not easy!
Recall that any associative algebra $A$ with product $ab$ can be made into a Lie algebra, denoted by $A^-$, by defining $[a, b] = ab - ba$ and into a Jordan algebra, denoted by $A^+$, by defining $a \circ b = (ab + ba)/2$

**Trivial Exercise**

A derivation is also a Lie derivation and a Jordan derivation.

**Theorem A**

Every Jordan derivation on $A^+$ is a derivation of $A$ ($A = M_n(\mathbb{R})$)

**Example**

There is a Lie derivation which is not a derivation ($A = M_n(\mathbb{R})$, namely $\delta(x) = \text{trace}(x) I$)

**Theorem B**

Every Lie derivation on $A^-$ is the sum of a derivation and a linear operator of the above form ($A = M_n(\mathbb{R})$)
THE SET $M_n(\mathbb{R})$ of $n$ by $n$ MATRICES IS AN ALGEBRA UNDER
MATRIX ADDITION $A + B$
AND MATRIX MULTIPLICATION $A \times B$
WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

For the Record: (square matrices)

$$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \quad [a_{ij}] \times [b_{ij}] = \left[ \sum_{k=1}^{n} a_{ik} b_{kj} \right]$$

DEFINITION

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO MATRIX MULTIPLICATION
IS A LINEAR PROCESS $\delta$: $\delta(A + B) = \delta(A) + \delta(B)$
WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \times B) = \delta(A) \times B + A \times \delta(B)$$
PROPOSITION

FIX A MATRIX $A$ in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$  

THEN $\delta_A$ IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH ARE CALLED INNER DERIVATIONS)

THEOREM (Noether, Wedderburn, Hochschild, Jacobson, Kaplansky, Kadison, Sakai, . . .)

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS INNER, THAT IS, OF THE FORM $\delta_A$ FOR SOME $A$ IN $M_n(\mathbb{R})$.

We gave a proof of this theorem for $n = 2$ in part 8 of series 1.
In 1990, Kadison and Larson and Sourour introduced the concept of local derivation for Banach algebras and modules. A linear mapping \( T : A \to A \), where \( A \) is an algebra is said to be a local derivation if for every \( a \) in \( A \) there exists a derivation \( D_a : A \to A \), depending on \( a \), satisfying \( T(a) = D_a(a) \).

**Theorem 1 (Kadison 1990, Larson and Sourour 1990)**

A local derivation on \( M_n(\mathbb{R}) \) is a derivation

**Example**

Let \( \mathbb{R}(x) \) denote the algebra of all rational functions (quotients of polynomials). There exists a local derivation of \( \mathbb{R}(x) \) which is not a derivation.

**Proof of Theorem 1**

(Ayupov, Kudaybergenov, Peralta 2014)
Steps in the Proof

• Every matrix $A \in M_n(\mathbb{C})$ is a (complex) linear combination of pairwise orthogonal projections, where a projection is a matrix $P$ such that $P = P^* = P^2$ and two projections are orthogonal if their product is 0.

• If $T$ is a local derivation on $M_n(\mathbb{C})$, then for each pair of orthogonal projections $P$ and $Q$, we have

$$QT(P) + T(Q)P = 0.$$  

• If $A = \sum \lambda_i P_i$ with $P_iP_j = 0$ for $i \neq j$, then

$$T(A^2) = AT(A) + T(A)A$$

• $T$ is a Jordan derivation, hence a derivation. Q.E.D.
Example

Let \( \mathbb{R}(x) \) denote the algebra of all rational functions (quotients of polynomials). There exists a local derivation of \( \mathbb{R}(x) \) which is not a derivation.

Exercise 1

The derivations of \( \mathbb{R}(x) \) are the mappings of the form \( f \mapsto gf' \) for some \( g \) in \( \mathbb{R}(x) \), where \( f' \) is the usual derivative of \( f \).

Exercise 2

The local derivations of \( \mathbb{R}(x) \) are the mappings which annihilate the constants.

Exercise 3

Write \( \mathbb{R}(x) = S + T \) where \( S \) is the 2-dimensional space generated by 1 and \( x \). Define \( \alpha : \mathbb{R}(x) \to \mathbb{R}(x) \) by \( \alpha(a + b) = b \). Then \( \alpha \) is a local derivation which is not a derivation.
The algebra is \( \mathbb{C}(x) \), the rational functions in the variable \( x \) over \( \mathbb{C} \). Let \( \mathbb{C}[x] \) be the subalgebra of polynomials. We note certain facts.

a. The derivations of \( \mathbb{C}(x) \) into itself are mappings of the form \( f \to gf' \) for some \( g \) in \( \mathbb{C}(x) \), where \( f' \) is the usual derivative of \( f \). Such a mapping is a derivation of \( \mathbb{C}(x) \). Let \( \delta \) be a derivation of \( \mathbb{C}(x) \) into \( \mathbb{C}(x) \) and let \( g \) be \( \delta(x) \). If \( p \in \mathbb{C}[x] \), then \( \delta(p) = gp' \) (applying the multiplicative property of the derivation). At the same time, if \( p \neq 0 \), then

\[
0 = \delta(1) = \delta(pp^{-1}) = \delta(p) p^{-1} + p \delta(p^{-1}),
\]

whence \( \delta(p^{-1}) = -\delta(p) p^{-2} = -gp'p^{-2} \). Thus, with \( p \) and \( q \) in \( \mathbb{C}[x] \),

\[
\delta(pq^{-1}) = \delta(p) q^{-1} + p \delta(q^{-1}) = gp'q^{-1} - gpq'q^{-2} = g[p'q - pq']q^{-2} = g[pq^{-1}]'.
\]

b. The local derivations of \( \mathbb{C}(x) \) are the linear mappings that annihilate the constants. If \( \alpha \) is a local derivation, then for each \( c \) in \( \mathbb{C} \), there is a derivation \( \delta \) of \( \mathbb{C}(x) \) such that \( \alpha(c) = \delta(c) = 0 \). Suppose, now, that \( \alpha \) is a linear mapping of \( \mathbb{C}(x) \) into \( \mathbb{C}(x) \) that annihilates the constants. Of course \( \alpha \) agrees with every derivation on all constants. If \( f \), in \( \mathbb{C}(x) \), is not a constant, then \( f' \neq 0 \). Let \( \delta(h) \) be \( (\alpha(f)/f')h' \). Then \( \delta \) is a derivation of \( \mathbb{C}(x) \) into \( \mathbb{C}(x) \), and \( \delta(f) = \alpha(f) \). Thus \( \alpha \) is a local derivation of \( \mathbb{C}(x) \).

c. We display a local derivation of \( \mathbb{C}(x) \) into itself that is not a derivation. With \( \mathbb{C}(x) \) considered as a vector space over \( \mathbb{C} \), the 2-dimensional subspace \( X \) generated by \( 1 \) and \( x \) has a complement \( Y \). Let \( x \) be the projection of \( \mathbb{C}(x) \) on \( Y \) along \( X \). Then \( x \) annihilates the constants, whence \( x \) is a local derivation from \( b \). If \( x \) were a derivation, then from \( a \), \( \alpha(f) \) would be \( \alpha(x)f' \), which is a \( 0 \) since \( \alpha(x) = 0 \). As \( \alpha \neq 0 \), \( \alpha \) is not a derivation.
A (not necessarily linear) mapping $T$ from an algebra $A$ into itself is said to be a 2-local derivation if for every $a, b \in A$ there exists a (of course linear) derivation $D_{a,b} : A \to A$, depending on $a$ and $b$, such that $D_{a,b}(a) = T(a)$ and $D_{a,b}(b) = T(b)$.

**Theorem 2**

Every 2-local derivation $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ (no linearity of $T$ is assumed) is a derivation.

Semrl 1997  
Kim-Kim 2004  
Ayupov-Kudaybergenov-Nuranjov 2014  
Ayupov-Arzikulov 2014  
Ayupov-Kudaybergenov 2015

**Proof of Theorem 2**

Kim, Kim 2004
Steps in the Proof

• Let $e_1, e_2$ be the standard basis for $\mathbb{C}^2$ and let

$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

• Any $S \in M_2(\mathbb{C})$ which commutes with $A$ is diagonal and any $U \in M_2(\mathbb{C})$ which commutes with $N$ has the form

$$U = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

• If $T$ is a 2-local derivation, then $T - D_{A,N}$ is another 2-local derivation which vanishes at $A$ and at $N$.

• Let $\phi = T - D_{A,N}$. Calculate that $\phi(E_{ij}) = 0$, where $E_{ij}$ are the matrix units

• Calculate that $E_{ij}\phi(X)E_{ij} = 0$ for all $X \in M_2(\mathbb{C})$. Thus $\phi(X) = 0$ for all $X$. Q.E.D.
Example (Zhang-Li 2006)

Let us consider the algebra of all upper-triangular complex $2 \times 2$-matrices

$$A = \left\{ x = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ 0 & \lambda_{22} \end{pmatrix} : \lambda_{ij} \in \mathbb{C} \right\}.$$

Define an operator $\Delta$ on $A$ by

$$\Delta(x) = \begin{cases} 0, & \text{if } \lambda_{11} \neq \lambda_{22}, \\ \begin{pmatrix} 0 & 2\lambda_{12} \\ 0 & 0 \end{pmatrix}, & \text{if } \lambda_{11} = \lambda_{22}. \end{cases}$$

Then $\Delta$ is a 2-local derivation, which is not a derivation.
THE CIRCLE PRODUCT ON THE SET OF MATRICES

**DEFINITION**

THE **CIRCLE PRODUCT** ON THE SET $M_n(\mathbb{R})$ OF MATRICES IS DEFINED BY

$$X \circ Y = \frac{(X \times Y + Y \times X)}{2}$$

THE SET $M_n(\mathbb{R})$ of $n$ by $n$ MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND CIRCLE MULTIPLICATION, WHICH IS COMMUTATIVE BUT NOT ASSOCIATIVE.

**DEFINITION**

A **DERIVATION** ON $M_n(\mathbb{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS A LINEAR PROCESS $\delta$ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$
PROPOSITION

FIX A MATRIX $A$ in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$ 

THEN $\delta_A$ IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION (ALSO CALLED AN INNER DERIVATION IN THIS CONTEXT$^a$)

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$^a$However, see the following remark. Also see some of the exercises (Dr. Gradus Ad Parnassum) in part 1 of these lectures

THEOREM (1972-Sinclair)

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS INNER, THAT IS, OF THE FORM $\delta_A$ FOR SOME $A$ IN $M_n(\mathbb{R})$.

REMARK (1937-Jacobson)

THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF SYMMETRIC MATRICES, FOR EXAMPLE.
### Table 2  $M_n(\mathbb{R})$ (ALGEBRAS)

<table>
<thead>
<tr>
<th>matrix</th>
<th>bracket</th>
<th>circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ab = a \times b$</td>
<td>$[a, b] = ab - ba$</td>
<td>$a \circ b = ab + ba$</td>
</tr>
<tr>
<td>Associative</td>
<td>Lie</td>
<td>Jordan</td>
</tr>
<tr>
<td>$\delta_a(x)$</td>
<td>$\delta_a(x)$</td>
<td>$\delta_a(x)$</td>
</tr>
<tr>
<td>$ax - xa$</td>
<td>$ax - xa$</td>
<td>$ax - xa$</td>
</tr>
<tr>
<td>or trace$(x)$</td>
<td></td>
<td></td>
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</tbody>
</table>
A linear mapping $T : A \rightarrow A$, where $A$ is a Jordan algebra is said to be a *local Jordan derivation* if for every $a$ in $A$ there exists a Jordan derivation $D_a : A \rightarrow A$, depending on $a$, satisfying $T(a) = D_a(a)$.

**Theorem 3**
A local Jordan derivation on $M_n(\mathbb{R})$ is a Jordan derivation

**Proof of Theorem 3**
Combine Theorem 1 with Theorem A.
A (not necessarily linear) mapping $T$ from a Jordan algebra $A$ into itself is said to be a 2-local Jordan derivation if for every $a, b \in A$ there exists a (linear) Jordan derivation $D_{a,b} : A \to A$, depending on $a$ and $b$, such that $D_{a,b}(a) = T(a)$ and $D_{a,b}(b) = T(b)$.

**Theorem 4**

Every 2-local Jordan derivation $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ (no linearity of $T$ is assumed) is a Jordan derivation.

**Proof of Theorem 4**

Combine Theorem 2 with Theorem A.
THE BRACKET PRODUCT ON THE SET OF MATRICES

**DEFINITION**

THE BRACKET PRODUCT ON THE SET $M_n(\mathbb{R})$ OF MATRICES IS DEFINED BY

$$[X, Y] = X \times Y - Y \times X$$

THE SET $M_n(\mathbb{R})$ OF $n$ by $n$ MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND BRACKET MULTIPLICATION, WHICH IS NOT ASSOCIATIVE AND NOT COMMUTATIVE.

**DEFINITION**

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS A LINEAR PROCESS $\delta$ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)].$$
PROPOSITION

FIX A MATRIX $A$ in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = [A, X] = A \times X - X \times A.$$  

THEN $\delta_A$ IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION (STILL CALLED INNER DERIVATION).

THEOREM

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS INNER, THAT IS, OF THE FORM $\delta_A$ FOR SOME $A$ IN $M_n(\mathbb{R})$.  

Full disclosure: this is actually not true. Check that the map $X \mapsto (\text{trace of } X)I$ is a derivation which is not inner ($I$ is the identity matrix). The correct statement is that every derivation of a semisimple finite dimensional Lie algebra is inner. $M_n(\mathbb{R})$ is a semisimple associative algebra under matrix multiplication, a semisimple Jordan algebra under circle multiplication, but not a semisimple Lie algebra under bracket multiplication. Please ignore this footnote until you find out what semisimple means in each context.
Theorem C (Equivalent definitions of semisimplicity)

Let \( \mathfrak{g} \) be a Lie algebra. Then the following are equivalent:

1. \( \mathfrak{g} \) does not contain any non-trivial solvable ideal.
2. \( \mathfrak{g} \) does not contain any non-trivial abelian ideal.
3. The Killing form \( K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \), defined as the bilinear form
   \[ K(x, y) := \text{tr}_\mathfrak{g}((\text{ad}x)(\text{ad}y)) \], is non-degenerate on \( \mathfrak{g} \).
4. \( \mathfrak{g} \) is isomorphic to the direct sum of finitely many non-abelian simple Lie algebras.
Theorem D (Classification of simple Lie algebras)

Up to isomorphism, every simple Lie algebra is of one of the following forms:

1. $A_n = \mathfrak{sl}_{n+1}$ for some $n \geq 1$. (trace zero, $\text{tr}(A) = 0$)
2. $B_n = \mathfrak{so}_{2n+1}$ for some $n \geq 2$. (skew-symmetric, $A^t = -A$)
3. $C_n = \mathfrak{sp}_{2n}$ for some $n \geq 3$. ($A^t J = -JA$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$)
4. $D_n = \mathfrak{so}_{2n}$ for some $n \geq 4$. (skew-symmetric)
5. $E_6$, $E_7$, or $E_8$.
7. $G_2$. 
A linear mapping $T : A \to A$, where $A$ is a Lie algebra is said to be a *local Lie derivation* if for every $a$ in $A$ there exists a Lie derivation $D_a : A \to A$, depending on $a$, satisfying $T(a) = D_a(a)$.

**Theorem 5**

A local Lie derivation on a finite dimensional semisimple Lie algebra (not $M_n(\mathbb{R})$!) is a Lie derivation

**Proof of Theorem 5**

Ayupov, Kudaybergenov 2014. This proof involves a detailed structure theory of semisimple Lie algebras and is beyond the scope of today’s talk.
A (not necessarily linear) mapping $T$ from a Lie algebra $A$ into itself is said to be a 2-local Lie derivation if for every $a, b \in A$ there exists a (linear) Lie derivation $D_{a,b} : A \to A$, depending on $a$ and $b$, such that $D_{a,b}(a) = T(a)$ and $D_{a,b}(b) = T(b)$.

**Theorem 6**

Every 2-local Lie derivation on a finite dimensional semisimple Lie algebra (no linearity is assumed) is a Lie derivation (not $M_n(\mathbb{R})$!).

**Proof of Theorem 6**

It suffices to prove that $T$ is linear, then Theorem 5 applies.