DERIVATIONS
An introduction to non associative algebra
(or, Playing havoc with the product rule)

Series 2—Part 4
Derivations, local derivations, and 2-local derivations on (algebras of) matrices

Colloquium
Fullerton College

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HISTORY

Series 1

- PART I FEBRUARY 8, 2011 **ALGEBRAS; DERIVATIONS**
- PART II JULY 21, 2011 **TRIPLE SYSTEMS; DERIVATIONS**
- PART III FEBRUARY 28, 2012 **MODULES; DERIVATIONS**
- PART IV JULY 26, 2012 **COHOMOLOGY (ASSOCIATIVE ALGEBRAS)**
- PART V OCTOBER 25, 2012 **THE SECOND COHOMOLOGY GROUP**
- PART VI MARCH 7, 2013 **COHOMOLOGY (LIE ALGEBRAS)**
- PART VII JULY 25, 2013 **COHOMOLOGY (JORDAN ALGEBRAS)**
- PART VIII SEPTEMBER 17, 2013 **VANISHING THEOREMS IN DIMENSIONS 1 AND 2 (ASSOCIATIVE ALGEBRAS)**
- PART IX FEBRUARY 18, 2014 **VANISHING THEOREMS IN DIMENSIONS 1 AND 2 (JORDAN ALGEBRAS)**

Series 2

- PART I JULY 24, 2014 **THE REMARKABLE CONNECTION BETWEEN JORDAN ALGEBRAS AND LIE ALGEBRAS (Two theorems relating different types of derivations)**
- PART II NOVEMBER 18, 2014 **THE REMARKABLE CONNECTION BETWEEN JORDAN ALGEBRAS AND LIE ALGEBRAS (Two theorems embedding triple systems in Lie algebras**
- (digression) FEBRUARY 24, 2015 **GENETIC ALGEBRAS**
- PART III JULY 15, 2015 **LOCAL DERIVATIONS**
- (Fall 2015 missed due to the flu)
- PART IV (today) FEBRUARY 23, 2016 **2-LOCAL DERIVATIONS**

"Slides" for all series 1 and series 2 talks available at

http://www.math.uci.edu/INSERT a "$\sim$" HERE brusso/undergraduate.html

ABSTRACT for February 23, 2016

After reviewing derivations and local derivations we consider local and 2-local derivations on the set of 2 by 2 matrices, considered
(1) as an associative algebra under matrix multiplication;
(2) as a Lie algebra under bracket multiplication;
(3) as a Jordan algebra under circle multiplication

ABSTRACT for July 15, 2015 (for context)

A local derivation is a linear map T of an algebra which at each element of the algebra agrees with some derivation D, which can vary from element to element. Every derivation is obviously a local derivation but not all local derivations are derivations. For a large class of operator algebras, the two notions coincide. Similar definitions and results hold for local derivations on triple systems. There is also a nonlinear version on algebras and triple systems.

Review of Algebras—Axiomatic approach

AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE OVER A FIELD) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION—we are downplaying multiplication by scalars (=numbers=field elements)

ADDITION IS DENOTED BY $a + b$ AND IS REQUIRED TO BE
COMMUTATIVE $\quad a + b = b + a$
AND ASSOCIATIVE $\quad (a + b) + c = a + (b + c)$

MULTIPLICATION IS DENOTED BY ab AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION
$(a + b)c = ac + bc, \quad a(b + c) = ab + ac$

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)
(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

Table 1 (FASHIONABLE) ALGEBRAS

commutative algebras $ab = ba$
(Real numbers, Complex numbers, Continuous functions)

associative algebras $a(bc) = (ab)c$
(Matrix multiplication)

Lie algebras $a^2 = 0$, $(ab)c + (bc)a + (ca)b = 0$
(Bracket multiplication on associative algebras: $[x, y] = xy - yx$)

Jordan algebras $ab = ba$, $a(a^2b) = a^2(ab)$
(Circle multiplication on associative algebras: $x \circ y = (xy + yx)/2$)

Types of derivations

Derivation

$\delta(ab) = a\delta(b) + \delta(a)b$

Lie derivation

$\delta([a,b]) = [a,\delta(b)] + [\delta(a),b]$

Jordan derivation

$\delta(a \circ b) = a \circ \delta(b) + \delta(a) \circ b$

There are two theorems relating different types of derivations. We will use one of them extensively today (Theorem A on the next page). We gave the proofs of Theorems A and B in Series 2, part I (July 24, 2014). The proofs are not easy!

Recall that any associative algebra A with product ab can be made into a Lie algebra, denoted by A^-, by defining $[a, b] = ab - ba$ and into a Jordan algebra, denoted by A^+, by defining $a \circ b = (ab + ba)/2$

Trivial Exercise

A derivation is also a Lie derivation and a Jordan derivation.

Theorem A

Every Jordan derivation on A^+ is a derivation of A ($A = M_n(\mathbb{R})$)

Example

There is a Lie derivation which is not a derivation ($A = M_n(\mathbb{R})$, namely $\delta(x) = trace(x)I$

Theorem B

Every Lie derivation on A^- is the sum of a derivation and a linear operator of the above form ($A = M_n(\mathbb{R})$)

DERIVATIONS ON MATRIX ALGEBRAS

THE SET $M_n(\mathbb{R})$ of n by n MATRICES IS AN ALGEBRA UNDER
MATRIX ADDITION $A + B$
AND **MATRIX MULTIPLICATION** $A \times B$
WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

For the Record: (square matrices)

$[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$ $\qquad$ $[a_{ij}] \times [b_{ij}] = [\sum_{k=1}^{n} a_{ik} b_{kj}]$

DEFINITION

A <u>DERIVATION</u> ON $M_n(\mathbb{R})$ WITH <u>RESPECT TO MATRIX MULTIPLICATION</u>
IS A LINEAR PROCESS δ: $\quad \delta(A + B) = \delta(A) + \delta(B)$
WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \times B) = \delta(A) \times B + A \times \delta(B)$$

.

PROPOSITION

FIX A MATRIX A in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH ARE CALLED **INNER DERIVATIONS**)

THEOREM (Noether,Wedderburn,Hochschild,Jacobson, Kaplansky,Kadison,Sakai,. . .)

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS INNER, THAT IS, OF THE FORM δ_A FOR SOME A IN $M_n(\mathbb{R})$.

We gave a proof of this theorem for $n = 2$ in part 8 of series 1.

In 1990, Kadison and Larson and Sourour introduced the concept of local derivation for Banach algebras and modules. A linear mapping $T : A \to A$, where A is an algebra is said to be a *local derivation* if for every a in A there exists a derivation $D_a : A \to A$, depending on a, satisfying $T(a) = D_a(a)$.

Theorem 1 (Kadison 1990, Larson and Sourour 1990)

A local derivation on $M_n(\mathbb{R})$ is a derivation

Example

Let $\mathbb{R}(x)$ denote the algebra of all rational functions (quotients of polynomials). There exists a local derivation of $\mathbb{R}(x)$ which is not a derivation.

Proof of Theorem 1

(Ayupov,Kudaybergenov,Peralta 2014)

Steps in the Proof

• Every matrix $A \in M_n(\mathbb{C})$ is a (complex) linear combination of pairwise orthogonal projections, where a projection is a matrix P such that $P = P^* = P^2$ and two projections are orthogonal if their product is 0.

• It T is a local derivation on $M_n(\mathbb{C})$, then for each pair of orthogonal projections P and Q, we have

$$QT(P) + T(Q)P = 0.$$

• If $A = \sum \lambda_i P_i$ with $P_i P_j = 0$ for $i \neq j$, then

$$T(A^2) = AT(A) + T(A)A$$

• T is a Jordan derivation, hence a derivation. Q.E.D.

Details of the example

From Series 2, part I (July 15, 2015)

Example

Let $\mathbb{R}(x)$ denote the algebra of all rational functions (quotients of polynomials). There exists a local derivation of $\mathbb{R}(x)$ which is not a derivation.

Exercise 1

The derivations of $\mathbb{R}(x)$ are the mappings of the form $f \mapsto gf'$ for some g in $\mathbb{R}(x)$, where f' is the usual derivative of f.

Exercise 2

The local derivations of $\mathbb{R}(x)$ are the mappings which annihilate the constants.

Exercise 3

Write $\mathbb{R}(x) = S + T$ where S is the 2-dimensional space generated by 1 and x. Define $\alpha : \mathbb{R}(x) \to \mathbb{R}(x)$ by $\alpha(a + b) = b$. Then α is a local derivation which is not a derivation.

The algebra is $\mathbb{C}(x)$, the rational functions in the variable x over $\mathbb{C}$. Let $\mathbb{C}[x]$ be the subalgebra of polynomials. We note certain facts.

a. The derivations of $\mathbb{C}(x)$ into itself are mappings of the form $f \to gf'$ for some g in $\mathbb{C}(x)$, where f' is the usual derivative of f. Such a mapping is a derivation of $\mathbb{C}(x)$. Let δ be a derivation of $\mathbb{C}(x)$ into $\mathbb{C}(x)$ and let g be $\delta(x)$. If $p \in \mathbb{C}[x]$, then $\delta(p) = gp'$ (applying the multiplicative property of the derivation). At the same time, if $p \neq 0$, then

$$0 = \delta(1) = \delta(pp^{-1}) = \delta(p)\, p^{-1} + p\delta(p^{-1}),$$

whence $\delta(p^{-1}) = -\delta(p)\, p^{-2} = -gp'p^{-2}$. Thus, with p and q in $\mathbb{C}[x]$,

$$\begin{aligned}\delta(pq^{-1}) &= \delta(p)\, q^{-1} + p\delta(q^{-1}) = gp'q^{-1} - gpq'q^{-2} \\ &= g[p'q - pq']\, q^{-2} = g[pq^{-1}]'.\end{aligned}$$

b. The local derivations of $\mathbb{C}(x)$ are the linear mappings that annihilate the constants. If α is a local derivation, then for each c in $\mathbb{C}$, there is a derivation δ of $\mathbb{C}(x)$ such that $\alpha(c) = \delta(c) = 0$. Suppose, now, that α is a linear mapping of $\mathbb{C}(x)$ into $\mathbb{C}(x)$ that annihilates the constants. Of course α agrees with every derivation on all constants. If f, in $\mathbb{C}(x)$, is not a constant, then $f' \neq 0$. Let $\delta(h)$ be $(\alpha(f)/f')\, h'$. Then δ is a derivation of $\mathbb{C}(x)$ into $\mathbb{C}(x)$, and $\delta(f) = \alpha(f)$. Thus α is a local derivation of $\mathbb{C}(x)$.

c. We display a local derivation of $\mathbb{C}(x)$ into itself that is not a derivation. With $\mathbb{C}(x)$ considered as a vector space over $\mathbb{C}$, the 2-dimensional subspace X generated by 1 and x has a complement Y. Let α be the projection of $\mathbb{C}(x)$ on Y along X. Then α annihilates the constants, whence α is a local derivation from b. If α were a derivation, then from a, $\alpha(f)$ would be $\alpha(x) f'$, which is a 0 since $\alpha(x) = 0$. As $\alpha \neq 0$, α is not a derivation.

A (not necessarily linear) mapping T from an algebra A into itself is said to be a *2-local derivation* if for every $a, b \in A$ there exists a (of course linear) derivation $D_{a,b} : A \to A$, depending on a and b, such that $D_{a,b}(a) = T(a)$ and $D_{a,b}(b) = T(b)$.

Theorem 2

Every 2-local derivation $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ (no linearity of T is assumed) is a derivation.

Semrl 1997
Kim-Kim 2004
Ayupov-Kudaybergenov-Nuranjov 2014
Ayupov-Arzikulov 2014
Ayupov-Kudaybergenov 2015

Proof of Theorem 2

Kim, Kim 2004

Steps in the Proof

• Let e_1, e_2 be the standard basis for $\mathbb{C}^2$ and let

$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

• Any $S \in M_2(\mathbb{C})$ which commutes with A is diagonal and any $U \in M_2(\mathbb{C})$ which commutes with N has the form

$$U = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

• If T is a 2-local derivation, then $T - D_{A,N}$ is another 2-local derivation which vanishes at A and at N.

• Let $\phi = T - D_{A,N}$. Calculate that $\phi(E_{ij}) = 0$, where E_{ij} are the matrix units

• Calculate that $E_{ij}\phi(X)E_{ij} = 0$ for all $X \in M_2(\mathbb{C})$. Thus $\phi(X) = 0$ for all X. Q.E.D.

Example (Zhang-Li 2006)

Let us consider the algebra of all upper-triangular complex 2×2-matrices

$$A = \left\{ x = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ 0 & \lambda_{22} \end{pmatrix} : \lambda_{ij} \in \mathbb{C} \right\}.$$

Define an operator Δ on A by

$$\Delta(x) = \begin{cases} 0, & \text{if } \lambda_{11} \neq \lambda_{22}, \\ \begin{pmatrix} 0 & 2\lambda_{12} \\ 0 & 0 \end{pmatrix}, & \text{if } \lambda_{11} = \lambda_{22}. \end{cases}$$

Then Δ is a 2-local derivation, which is not a derivation.

THE CIRCLE PRODUCT ON THE SET OF MATRICES

DEFINITION

THE **CIRCLE PRODUCT** ON THE SET $M_n(\mathbb{R})$ OF MATRICES IS DEFINED BY

$$X \circ Y = (X \times Y + Y \times X)/2$$

THE SET $M_n(\mathbb{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND CIRCLE MULTIPLICATION, WHICH IS COMMUTATIVE BUT NOT ASSOCIATIVE.

DEFINITION

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

PROPOSITION

FIX A MATRIX A in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION (ALSO CALLED AN INNER DERIVATION IN THIS CONTEXT[a])

[a]However, see the following remark. Also see some of the exercises (Dr. Gradus Ad Parnassum) in part 1 of these lectures

THEOREM (1972-Sinclair)

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS INNER, THAT IS, OF THE FORM δ_A FOR SOME A IN $M_n(\mathbb{R})$.

REMARK (1937-Jacobson)
THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF SYMMETRIC MATRICES, FOR EXAMPLE.

Table 2 $M_n(\mathbb{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
Associative	Lie	Jordan
$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$
	or trace$(x)I$	

A linear mapping $T : A \to A$, where A is a Jordan algebra is said to be a *local Jordan derivation* if for every a in A there exists a Jordan derivation $D_a : A \to A$, depending on a, satisfying $T(a) = D_a(a)$.

Theorem 3

A local Jordan derivation on $M_n(\mathbb{R})$ is a Jordan derivation

Proof of Theorem 3

Combine Theorem 1 with Theorem A.

A (not necessarily linear) mapping T from a Jordan algebra A into itself is said to be a *2-local Jordan derivation* if for every $a, b \in A$ there exists a (linear) Jordan derivation $D_{a,b} : A \to A$, depending on a and b, such that $D_{a,b}(a) = T(a)$ and $D_{a,b}(b) = T(b)$.

Theorem 4

Every 2-local Jordan derivation $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ (no linearity of T is assumed) is a Jordan derivation.

Proof of Theorem 4

Combine Theorem 2 with Theorem A.

THE BRACKET PRODUCT ON THE SET OF MATRICES

DEFINITION

THE **BRACKET PRODUCT** ON THE SET $M_n(\mathbb{R})$ OF MATRICES IS DEFINED BY

$$[X, Y] = X \times Y - Y \times X$$

THE SET $M_n(\mathbb{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND BRACKET MULTIPLICATION, WHICH IS NOT ASSOCIATIVE AND NOT COMMUTATIVE.

DEFINITION

A DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)].$$

PROPOSITION

FIX A MATRIX A in $M_n(\mathbb{R})$ AND DEFINE

$$\delta_A(X) = [A, X] = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION (STILL CALLED **INNER DERIVATION**).

THEOREM

EVERY DERIVATION ON $M_n(\mathbb{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS INNER, THAT IS, OF THE FORM δ_A FOR SOME A IN $M_n(\mathbb{R})$. [a]

[a] Full disclosure: this is actually not true. Check that the map $X \mapsto (\text{trace of } X)I$ is a derivation which is not inner (I is the identity matrix). The correct statement is that every derivation of a semisimple finite dimensional Lie algebra is inner. $M_n(\mathbb{R})$ is a semisimple associative algebra under matrix multiplication, a semisimple Jordan algebra under circle multiplication, but not a semisimple Lie algebra under bracket multiplication. Please ignore this footnote until you find out what semisimple means in each context

Theorem C (Equivalent definitions of semisimplicity)

Let $\mathfrak{g}$ be a Lie algebra. Then the following are equivalent:

1. $\mathfrak{g}$ does not contain any non-trivial solvable ideal.
2. $\mathfrak{g}$ does not contain any non-trivial abelian ideal.
3. The Killing form $K : \mathfrak{g} \times \mathfrak{g} \to \mathbf{C}$, defined as the bilinear form $K(x, y) := \mathrm{tr}_{\mathfrak{g}}((\mathrm{ad}x)(\mathrm{ad}y))$, is non-degenerate on $\mathfrak{g}$.
4. $\mathfrak{g}$ is isomorphic to the direct sum of finitely many non-abelian simple Lie algebras.

Theorem D (Classification of simple Lie algebras)

Up to isomorphism, every simple Lie algebra is of one of the following forms:

1. $A_n = \mathfrak{sl}_{n+1}$ for some $n \geq 1$. (trace zero, $\mathrm{tr}\,(A) = 0$)
2. $B_n = \mathfrak{so}_{2n+1}$ for some $n \geq 2$. (skew-symmetric, $A^t = -A$)
3. $C_n = \mathfrak{sp}_{2n}$ for some $n \geq 3$. $\left(A^t J = -JA,\ J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$
4. $D_n = \mathfrak{so}_{2n}$ for some $n \geq 4$. (skew-symmetric)
5. $E_6, E_7, or E_8$.
6. F_4.
7. G_2.

A linear mapping $T : A \to A$, where A is a Lie algebra is said to be a *local Lie derivation* if for every a in A there exists a Lie derivation $D_a : A \to A$, depending on a, satisfying $T(a) = D_a(a)$.

Theorem 5

A local Lie derivation on a finite dimensional semisimple Lie algebra (not $M_n(\mathbb{R})$!) is a Lie derivation

Proof of Theorem 5

Ayupov,Kudaybergenov 2014. This proof involves a detailed structure theory of semisimple Lie algebras and is beyond the scope of today's talk.

A (not necessarily linear) mapping T from a Lie algebra A into itself is said to be a *2-local Lie derivation* if for every $a, b \in A$ there exists a (linear) Lie derivation $D_{a,b} : A \to A$, depending on a and b, such that $D_{a,b}(a) = T(a)$ and $D_{a,b}(b) = T(b)$.

Theorem 6

Every 2-local Lie derivation on a finite dimensional semisimple Lie algebra (no linearity is assumed) is a Lie derivation (not $M_n(\mathbb{R})$!).

Proof of Theorem 6

It suffices to prove that T is linear, then Theorem 5 applies.