DERIVATIONS

Introduction to non-associative algebra

OR

Playing havoc with the product rule?

PART VI—COHOMOLOGY OF LIE ALGEBRAS

BERNARD RUSSO

University of California, Irvine

FULLERTON COLLEGE
DEPARTMENT OF MATHEMATICS
MATHEMATICS COLLOQUIUM
MARCH 7, 2013

THIS PAGE PURPOSELY LEFT BLANK

HISTORY OF THESE LECTURES

PART I
ALGEBRAS
FEBRUARY 8, 2011

PART II
TRIPLE SYSTEMS
JULY 21, 2011

PART III
MODULES AND DERIVATIONS
FEBRUARY 28, 2012

PART IV
COHOMOLOGY OF ASSOCIATIVE ALGEBRAS
JULY 26, 2012

PART V
MEANING OF THE SECOND COHOMOLOGY GROUP
OCTOBER 25, 2012

PART VI
COHOMOLOGY OF LIE ALGEBRAS
MARCH 7, 2013

OUTLINE OF TODAY'S TALK

1. DERIVATIONS ON ALGEBRAS
(FROM FEBRUARY 8, 2011)

2. SET THEORY and GROUPS
(EQUIVALENCE CLASSES and QUOTIENT GROUPS)
(FROM OCTOBER 25, 2012)

3. FIRST COHOMOLOGY GROUP
(FROM JULY 26, 2012)

4. SECOND COHOMOLOGY GROUP

5. COHOMOLOGY OF LIE ALGEBRAS

PART I: REVIEW OF ALGEBRAS

AXIOMATIC APPROACH

AN ALGEBRA IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH TWO BINARY OPERATIONS, CALLED ADDITION AND MULTIPLICATION

ADDITION IS DENOTED BY

$$a + b$$

AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE

$$a + b = b + a, \quad (a + b) + c = a + (b + c)$$

MULTIPLICATION IS DENOTED BY

$$ab$$

AND IS REQUIRED TO BE DISTRIBUTIVE WITH RESPECT TO ADDITION

$$(a+b)c = ac+bc, \quad a(b+c) = ab+ac$$

AN ALGEBRA IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)

(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

Table 2

ALGEBRAS

commutative algebras

$$ab = ba$$

associative algebras

$$a(bc) = (ab)c$$

Lie algebras

$$a^2 = 0$$

$$(ab)c + (bc)a + (ca)b = 0$$

Jordan algebras

$$ab = ba$$

$$a(a^2b) = a^2(ab)$$

THIS PAGE PURPOSELY LEFT BLANK

DERIVATIONS ON THE SET OF MATRICES

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER

MATRIX ADDITION

$$A + B$$

AND

MATRIX MULTIPLICATION

$$A \times B$$

WHICH IS ASSOCIATIVE BUT NOT COMMUTATIVE.

DEFINITION 2

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \times B) = \delta(A) \times B + A \times \delta(B)$$

.

PROPOSITION 2

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION (WHICH CAN BE NON-ZERO)

THEOREM 2

(1942 Hochschild)

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO MATRIX MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

Gerhard Hochschild (1915–2010)

(Photo 1968)

Gerhard Paul Hochschild was an American mathematician who worked on Lie groups, algebraic groups, homological algebra and algebraic number theory.

THIS PAGE PURPOSELY LEFT BLANK

THE BRACKET PRODUCT ON THE SET OF MATRICES

THE BRACKET PRODUCT ON THE SET $M_n(\mathbf{R})$ OF MATRICES IS DEFINED BY

$$[X, Y] = X \times Y - Y \times X$$

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND BRACKET MULTIPLICATION, WHICH IS NOT ASSOCIATIVE AND NOT COMMUTATIVE.

DEFINITION 3

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO BRACKET MULTIPLICATION

IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A,B]) = [\delta(A),B] + [A,\delta(B)]$$

.

PROPOSITION 3

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = [A,X] = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION

THEOREM 3

(1942 Hochschild, Zassenhaus)
EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO BRACKET MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

Hans Zassenhaus (1912–1991)

Hans Julius Zassenhaus was a German mathematician, known for work in many parts of abstract algebra, and as a pioneer of computer algebra.

THIS PAGE PURPOSELY LEFT BLANK

THE CIRCLE PRODUCT ON THE SET OF MATRICES

THE CIRCLE PRODUCT ON THE SET $M_n(\mathbf{R})$ OF MATRICES IS DEFINED BY

$$X \circ Y = (X \times Y + Y \times X)/2$$

THE SET $M_n(\mathbf{R})$ of n by n MATRICES IS AN ALGEBRA UNDER MATRIX ADDITION AND CIRCLE MULTIPLICATION, WHICH IS COMMUTATIVE BUT NOT ASSOCIATIVE.

DEFINITION 4

A DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION

IS A LINEAR PROCESS δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

PROPOSITION 4

FIX A MATRIX A in $M_n(\mathbf{R})$ AND DEFINE

$$\delta_A(X) = A \times X - X \times A.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION

THEOREM 4

(1972-Sinclair)

EVERY DERIVATION ON $M_n(\mathbf{R})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{R})$.

REMARK

(1937-Jacobson)

THE ABOVE PROPOSITION AND THEOREM NEED TO BE MODIFIED FOR THE SUBALGEBRA (WITH RESPECT TO CIRCLE MULTIPLICATION) OF SYMMETRIC MATRICES.

Alan M. Sinclair (retired)

Nathan Jacobson (1910–1999)

Nathan Jacobson was an American mathematician who was recognized as one of the leading algebraists of his generation, and he was also famous for writing more than a dozen standard monographs.

Table 1

$M_n(\mathbf{R})$ (ALGEBRAS)

matrix	bracket	circle
$ab = a \times b$	$[a, b] = ab - ba$	$a \circ b = ab + ba$
Th. 2	Th.3	Th.4
$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$	$\delta_a(x) = ax - xa$

THIS PAGE PURPOSELY LEFT BLANK

PART 2 OF TODAY'S TALK

A **partition** of a set X is a disjoint class $\{X_i\}$ of non-empty subsets of X whose union is X

- $\{1, 2, 3, 4, 5\} = \{1, 3, 5\} \cup \{2, 4\}$
- $\{1, 2, 3, 4, 5\} = \{1\} \cup \{2\} \cup \{3, 5\} \cup \{4\}$
- $\mathbf{R} = \mathbf{Q} \cup (\mathbf{R} - \mathbf{Q})$
- $\mathbf{R} = \cdots \cup [-2, -1) \cup [-1, 0) \cup [0, 1) \cup \cdots$

A **binary relation** on the set X is a subset R of $X \times X$. For each ordered pair

$$(x, y) \in X \times X,$$

x is said to be related to y if $(x, y) \in R$.

- $R = \{(x, y) \in \mathbf{R} \times \mathbf{R} : x < y\}$
- $R = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y = \sin x\}$
- For a partition $X = \cup_i X_i$ of a set X, let $R = \{(x, y) \in X \times X : x, y \in X_i \text{ for some } i\}$

An **equivalence relation** on a set X is a relation $R \subset X \times X$ satisfying

reflexive $(x, x) \in R$
symmetric $(x, y) \in R \Rightarrow (y, x) \in R$
transitive $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$

There is a one to one correspondence between equivalence relations on a set X and partitions of that set.

NOTATION

- If R is an equivalence relation we denote $(x, y) \in R$ by $x \sim y$.
- The equivalence class containing x is denoted by $[x]$. Thus

$$[x] = \{y \in X : x \sim y\}.$$

EXAMPLES

- equality: $R = \{(x,x) : x \in X\}$
- equivalence class of fractions
  = rational number:

  $$R = \{(\frac{a}{b}, \frac{c}{d}) : a,b,c,d \in \mathbf{Z}, b \neq 0, d \neq 0, ad = bc\}$$
- equipotent sets: X and Y are equivalent if there exists a function $f : X \to Y$ which is one to one and onto.
- half open interval of length one:
  $R = \{(x,y) \in \mathbf{R} \times \mathbf{R} : x - y$ is an integer$\}$
- integers modulo n:

$R = \{(x,y) \in \mathbf{N} \times \mathbf{N} : x - y$ is divisible by $n\}$

THIS PAGE PURPOSELY LEFT BLANK

PART 2 OF TODAY'S TALK (continued)

A **group** is a set G together with an operation (called *multiplication*) which associates with each ordered pair x, y of elements of G a third element in G (called their *product* and written xy) in such a manner that

- multiplication is *associative*: $(xy)z = x(yz)$
- there exists an element e in G, called the *identity* element with the property that $$xe = ex = x \text{ for all } x$$
- to each element x, there corresponds another element in G, called the *inverse* of x and written x^{-1}, with the property that $$xx^{-1} = x^{-1}x = e$$

TYPES OF GROUPS

- commutative groups: $xy = yx$
- finite groups $\{g_1, g_2, \cdots, g_n\}$
- infinite groups $\{g_1, g_2, \cdots, g_n, \cdots\}$
- cyclic groups $\{e, a, a^2, a^3, \ldots\}$

EXAMPLES

1. $\mathbf{R}, +, 0, x^{-1} = -x$
2. positive real numbers, $\times, 1, x^{-1} = 1/x$
3. $\mathbf{R}^n$,vector addition,$(0, \cdots, 0)$, $(x_1, \cdots, x_n)^{-1} = (-x_1, \cdots, -x_n)$
4. $\mathcal{C}, +, 0, f^{-1} = -f$
5. $\{0, 1, 2, \cdots, m-1\}$, addition modulo m, 0, $k^{-1} = m - k$
6. permutations (=one to one onto functions), composition, identity permutation, inverse permutation
7. $M_n(\mathbf{R})$,+,0,$A^{-1} = [-a_{ij}]$
8. non-singular matrices, matrix multiplication, identity matrix, matrix inverse

Which of these are commutative, finite, infinite?

We shall consider only commutative groups and we shall denote the multiplication by $+$, the identity by 0, and inverse by -. No confusion should result.

ALERT

Counterintuitively, a very important (commutative) group is a group with one element

Let H be a subgroup of a commutative group G. That is, H is a subset of G and is a group under the same +,0,- as G.

Define an equivalence relations on G as follows: $x \sim y$ if $x - y \in H$.

The set of equivalence classes is a group under the definition of addition given by

$$[x] + [y] = [x + y].$$

This group is denoted by G/H and is called the **quotient group** of G by H.

Special cases:

$H = \{e\}$; $G/H = G$ (isomorphic)

$H = G$; $G/H = \{e\}$ (isomorphic)

EXAMPLES

1. $G = \mathbf{R}, +, 0, x^{-1} = -x$;

   $H = \mathbf{Z}$ or $H = \mathbf{Q}$

2. $\mathbf{R}^n$,vector addition,$(0, \cdots, 0)$,
   $(x_1, \cdots, x_n)^{-1} = (-x_1, \cdots, -x_n)$;

   $H = \mathbf{Z}^n$ or $H = \mathbf{Q}^n$

3. $\mathcal{C}, +, 0, f^{-1} = -f$;

   $H = \mathcal{D}$ or $H =$ polynomials

4. $M_n(\mathbf{R})$,+,0,$A^{-1} = [-a_{ij}]$;

   $H =$symmetric matrices, or
   $H =$anti-symmetric matrices

THIS PAGE PURPOSELY LEFT BLANK

PART 3 OF TODAY'S TALK

The basic formula of homological algebra

$$
\begin{gathered}
F(x_1, \ldots, x_n, x_{n+1}) = \\
x_1 f(x_2, \ldots, x_{n+1}) \\
-f(x_1 x_2, x_3, \ldots, x_{n+1}) \\
+f(x_1, x_2 x_3, x_4, \ldots, x_{n+1}) \\
- \cdots \\
\pm f(x_1, x_2, \ldots, x_n x_{n+1}) \\
\mp f(x_1, \ldots, x_n) x_{n+1}
\end{gathered}
$$

OBSERVATIONS

- n is a positive integer, $n = 1, 2, \cdots$
- f is a function of n variables
- F is a function of $n + 1$ variables
- $x_1, x_2, \cdots, x_{n+1}$ belong an algebra A
- $f(y_1, \ldots, y_n)$ and $F(y_1, \cdots, y_{n+1})$ also belong to A

HIERARCHY

- $x_1, x_2, \ldots, x_n$ are points (or vectors)
- f and F are functions—they take points to points
- T, defined by $T(f) = F$ is a transformation—takes functions to functions

- points $x_1, \ldots, x_{n+1}$ and $f(y_1, \ldots, y_n)$ will belong to an algebra A
- functions f will be either <u>constant</u>, <u>linear</u> or <u>multilinear</u> (hence so will F)
- transformation T is linear

SHORT FORM OF THE FORMULA

$$(Tf)(x_1, \ldots, x_n, x_{n+1})$$

$$= x_1 f(x_2, \ldots, x_{n+1})$$

$$+ \sum_{j=1}^{n} (-1)^j f(x_1, \ldots, x_j x_{j+1}, \ldots, x_{n+1})$$

$$+(-1)^{n+1} f(x_1, \ldots, x_n) x_{n+1}$$

FIRST CASES

$\underline{n = 0}$

If f is any constant function from A to A, say, $f(x) = b$ for all x in A, where b is a fixed element of A, we have, consistent with the basic formula,

$$T_0(f)(x_1) = x_1 b - b x_1$$

$\underline{n = 1}$

If f is a linear map from A to A, then

$$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

$\underline{n = 2}$

If f is a bilinear map from $A \times A$ to A, then

$$T_2(f)(x_1, x_2, x_3) =$$

$$x_1 f(x_2, x_3) - f(x_1 x_2, x_3)$$

$$+ f(x_1, x_2 x_3) - f(x_1, x_2) x_3$$

Kernel and Image of a linear transformation

- $G : X \to Y$

  Since X and Y are vector spaces, they are in particular, commutative groups.

- **Kernel** of G (also called **nullspace** of G) is
  $\ker G = \{x \in X : G(x) = 0\}$

  This is a subgroup of X

- **Image** of G is
  $\operatorname{im} G = \{G(x) : x \in X\}$

  This is a subgroup of Y

What is the kernel of D on $\mathcal{D}$?

What is the image of D on $\mathcal{D}$?

(Hint: Second Fundamental theorem of calculus)

We now let $G = T_0, T_1, T_2$

$\underline{G = T_0}$

$X = A$ (the algebra)

$Y = L(A)$ (all linear transformations on A)

$$T_0(f)(x_1) = x_1 b - b x_1$$

$\ker T_0 = \{b \in A : xb - bx = 0$ for all $x \in A\}$
(center of A)

$\operatorname{im} T_0 =$ the set of all linear maps of A of the form $x \mapsto xb - bx$,

in other words, the set of all inner derivations of A

$\ker T_0$ is a subgroup of A

$\operatorname{im} T_0$ is a subgroup of $L(A)$

$\underline{G = T_1}$

$X = L(A)$ (linear transformations on A)

$Y = L^2(A)$ (bilinear transformations on $A \times A$)

$$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

$\ker T_1 = \{f \in L(A) : T_1 f(x_1, x_2) = 0 \text{ for all } x_1, x_2 \in A\}$ = the set of all derivations of A

$\operatorname{im} T_1$ = the set of all bilinear maps of $A \times A$ of the form

$$(x_1, x_2) \mapsto x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2,$$

for some linear function $f \in L(A)$.

$\ker T_1$ is a subgroup of $L(A)$

$\operatorname{im} T_1$ is a subgroup of $L^2(A)$

$$L^0(A) \xrightarrow{T_0} L(A) \xrightarrow{T_1} L^2(A) \xrightarrow{T_2} L^3(A) \cdots$$

FACTS:

- $T_1 \circ T_0 = 0$
- $T_2 \circ T_1 = 0$
- $\cdots$
- $T_{n+1} \circ T_n = 0$
- $\cdots$

Therefore

$$\text{im}\, T_n \subset \ker T_{n+1} \subset L^n(A)$$

and

im T_n is a subgroup of ker T_{n+1}

- $\operatorname{im} T_0 \subset \ker T_1$

  says

  Every inner derivation is a derivation

- $\operatorname{im} T_1 \subset \ker T_2$

  says

  for every linear map f, the bilinear map F defined by

  $$F(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

  satisfies the equation

  $$x_1 F(x_2, x_3) - F(x_1 x_2, x_3) +$$

  $$F(x_1, x_2 x_3) - F(x_1, x_2) x_3 = 0$$

  for every $x_1, x_2, x_3 \in A$.

The cohomology groups of A are defined as the quotient groups

$$H^n(A) = \frac{\ker T_n}{\operatorname{im} T_{n-1}}$$

$$(n = 1, 2, \ldots)$$

Thus

$$H^1(A) = \frac{\ker T_1}{\operatorname{im} T_0} = \frac{\text{derivations}}{\text{inner derivations}}$$

$$H^2(A) = \frac{\ker T_2}{\operatorname{im} T_1} = \frac{?}{?}$$

The theorem that every derivation of $M_n(\mathbf{R})$ is inner (that is, of the form δ_a for some $a \in M_n(\mathbf{R})$) can now be restated as:

"the cohomology group $H^1(M_n(\mathbf{R}))$ is the trivial one element group"

$\underline{G = T_2}$

$X = L^2(A)$ (bilinear transformations on $A \times A$)

$Y = L^3(A)$ (trilinear transformations on $A \times A \times A$)

$$T_2(f)(x_1, x_2, x_3) = x_1 f(x_2, x_3) - f(x_1 x_2, x_3) + f(x_1, x_2 x_3) - f(x_1, x_2) x_3$$

$\ker T_2 = \{f \in L(A) : T_2 f(x_1, x_2, x_3) = 0 \text{ for all } x_1, x_2, x_3 \in A\} =?$

$\operatorname{im} T_2 =$ the set of all trilinear maps h of $A \times A \times A$ of the form*

$$h(x_1, x_2, x_3) = x_1 f(x_2, x_3) - f(x_1 x_2, x_3) + f(x_1, x_2 x_3) - f(x_1, x_2) x_3,$$

for some bilinear function $f \in L^2(A)$.

$\ker T_2$ is a subgroup of $L^2(A)$

$\operatorname{im} T_2$ is a subgroup of $L^3(A)$

*we do not use $\operatorname{im} T_2$ in what follows

THIS PAGE PURPOSELY LEFT BLANK

PART 4 OF TODAY'S TALK

INTERPRETATION OF THE SECOND COHOMOLOGY GROUP (ASSOCIATIVE ALGEBRAS)

THIS PAGE PURPOSELY LEFT BLANK

Homomorphisms of groups

$f : G_1 \to G_2$ is a homomorphism if

$$f(x+y) = f(x) + f(y)$$

- $f(G_1)$ is a subgroup of G_2
- $\ker f$ is a subgroup of G_1
- $G_1/\ker f$ is isomorphic to $f(G_1)$

(isomorphism =
one to one and onto homomorphism)

Homomorphisms of algebras

$h : A_1 \to A_2$ is a homomorphism if

$$h(x+y) = h(x) + h(y)$$

and

$$h(xy) = h(x)h(y)$$

- $h(A_1)$ is a subalgebra of A_2
- $\ker h$ is a subalgebra of A_1 (actually, an ideal[†] in A_1)
- $A_1/\ker h$ is isomorphic to $h(A_1)$

(isomorphism = one to one and onto homomorphism)

[†]An **ideal** in an algebra A is a subalgebra I with the property that $AI \cup IA \subset I$, that is, $xa, ax \in I$ whenever $x \in I$ and $a \in A$

EXTENSIONS

Let A be an algebra. Let M be another algebra which contains an ideal I and let $g : M \to A$ be a homomorphism.

In symbols,

$$I \stackrel{\subsetneq}{\to} M \stackrel{g}{\to} A$$

This is called an **extension of A by I** if

- $\ker g = I$
- $\operatorname{im} g = A$

It follows that M/I is isomorphic to A

EXAMPLE 1

Let A be an algebra.

Define an algebra $M = A \oplus A$ to be the set $A \times A$ with addition

$$(a, x) + (b, y) = (a + b, x + y)$$

and product

$$(a, x)(b, y) = (ab, xy)$$

- $\{0\} \times A$ is an ideal in M
- $(\{0\} \times A)^2 \neq 0$
- $g : M \to A$ defined by $g(a, x) = a$ is a homomorphism
- M is an extension of $\{0\} \times A$ by A.

EXAMPLE 2

Let A be an algebra and let $h \in \ker T_2 \subset L^2(A)$.

Recall that this means that for all $x_1, x_2, x_3 \in A$,

$$x_1 f(x_2, x_3) - f(x_1 x_2, x_3)$$

$$+f(x_1, x_2 x_3) - f(x_1, x_2) x_3 = 0$$

Define an algebra M_h to be the set $A \times A$ with addition

$$(a, x) + (b, y) = (a + b, x + y)$$

and the product

$$(a, x)(b, y) = (ab, ay + xb + h(a, b))$$

Because $h \in \ker T_2$, this algebra is **ASSOCIATIVE!** whenever A is associative.

THE PLOT THICKENS

- $\{0\} \times A$ is an ideal in M_h
- $(\{0\} \times A)^2 = 0$
- $g : M_h \to A$ defined by $g(a, x) = a$ is a homomorphism
- M_h is an extension of $\{0\} \times A$ by A.

EQUIVALENCE OF EXTENSIONS

Extensions

$$I \stackrel{\subsetneq}{\to} M \stackrel{g}{\to} A$$

and

$$I \stackrel{\subsetneq}{\to} M' \stackrel{g'}{\to} A$$

are said to be equivalent if

there is an isomorphism $\psi : M \to M'$

such that

- $\psi(x) = x$ for all $x \in I$
- $g = g' \circ \psi$

(Is this an equivalence relation?)

EXAMPLE 2—continued

Let $h_1, h_2 \in \ker T_2$.

We then have two extensions of A by $\{0\} \times A$, namely

$$\{0\} \times A \overset{\subsetneq}{\to} M_{h_1} \overset{g_1}{\to} A$$

and

$$\{0\} \times A \overset{\subsetneq}{\to} M_{h_2} \overset{g_2}{\to} A$$

Now suppose that h_1 is equivalent‡ to h_2, $h_1 - h_2 = T_1 f$ for some $f \in L(A)$

- The above two extensions are equivalent.
- We thus have a mapping from $H^2(A, A)$ into the set of equivalence classes of extensions of A by the ideal $\{0\} \times A$

‡This is the same as saying that $[h_1] = [h_2]$ as elements of $H^2(A, A) = \ker T_2 / \operatorname{im} T_1$

THIS PAGE PURPOSELY LEFT BLANK

PART 5 OF TODAY'S TALK

COHOMOLOGY OF LIE ALGEBRAS

The basic formula of homological algebra (ASSOCIATIVE ALGEBRAS)

$$
\begin{gathered}
F(x_1, \ldots, x_n, x_{n+1}) = \\
x_1 f(x_2, \ldots, x_{n+1}) \\
-f(x_1 x_2, x_3, \ldots, x_{n+1}) \\
+f(x_1, x_2 x_3, x_4, \ldots, x_{n+1}) \\
- \cdots \\
\pm f(x_1, x_2, \ldots, x_n x_{n+1}) \\
\mp f(x_1, \ldots, x_n) x_{n+1}
\end{gathered}
$$

OBSERVATIONS

- n is a positive integer, $n = 1, 2, \cdots$
- f is a function of n variables
- F is a function of $n+1$ variables
- $x_1, x_2, \cdots, x_{n+1}$ belong an algebra A
- $f(y_1, \ldots, y_n)$ and $F(y_1, \cdots, y_{n+1})$ also belong to A

The basic formula of homological algebra (LIE ALGEBRAS)

$$F(x_1, \ldots, x_n, x_{n+1}) =$$
$$\pm[f(x_2, \ldots, x_{n+1}), x_1]$$
$$\mp[f(x_1, x_3, \ldots, x_{n+1}), x_2]$$
$$\ldots$$
$$+[f(x_1, x_2, \ldots, x_{n-1}, x_n), x_{n+1}]$$
$$+$$
$$-f(x_3, x_4, \ldots, x_{n+1}, [x_1, x_2])$$
$$+f(x_2, x_4, \ldots, x_{n+1}, [x_1, x_3])$$
$$-f(x_2, x_3, \ldots, x_{n+1}, [x_1, x_4])$$
$$\ldots$$
$$\pm f(x_2, x_3, \ldots, x_n, [x_1, x_{n+1}])$$
$$+$$
$$-f(x_1, x_4, \ldots, x_{n+1}, [x_2, x_3])$$
$$+f(x_1, x_3, \ldots, x_{n+1}, [x_2, x_4])$$
$$-f(x_1, x_3, \ldots, x_{n+1}, [x_2, x_5])$$
$$\ldots$$
$$\pm f(x_1, x_3, \ldots, x_n, [x_2, x_{n+1}])$$
$$+$$
$$\ldots$$
$$+$$
$$-f(x_1, x_2, \ldots, x_{n-1}, [x_n, x_{n+1}])$$

HIERARCHY
(ASSOCIATIVE ALGEBRAS)

- $x_1, x_2, \ldots, x_n$ are points (or vectors)
- f and F are functions—they take points to points
- T, defined by $T(f) = F$ is a transformation—takes functions to functions

- points $x_1, \ldots, x_{n+1}$ and $f(y_1, \ldots, y_n)$ will belong to an **ASSOCIATIVE** algebra A
- functions f will be either <u>constant</u>, <u>linear</u> or <u>multilinear</u> (hence so will F)
- transformation T is linear

HIERARCHY
(LIE ALGEBRAS)

- $x_1, x_2, \ldots, x_n$ are points (or vectors)
- f and F are functions—they take points to points
- T, defined by $T(f) = F$ is a transformation—takes functions to functions

- points $x_1, \ldots, x_{n+1}$ and $f(y_1, \ldots, y_n)$ will belong to a **LIE** algebra A
- functions f will be either constant, linear or **SKEW-SYMMETRIC** multilinear (hence so will F)
- transformation T is linear

SHORT FORM OF THE FORMULA (ASSOCIATIVE ALGEBRAS)

$$(Tf)(x_1, \ldots, x_n, x_{n+1})$$

$$= x_1 f(x_2, \ldots, x_{n+1})$$

$$+ \sum_{j=1}^{n} (-1)^j f(x_1, \ldots, x_j x_{j+1}, \ldots, x_{n+1})$$

$$+(-1)^{n+1} f(x_1, \ldots, x_n) x_{n+1}$$

FIRST CASES

$\underline{n = 0}$

If f is any constant function from A to A, say, $f(x) = b$ for all x in A, where b is a fixed element of A, we have, consistent with the basic formula,

$$T_0(f)(x_1) = x_1 b - b x_1$$

SHORT FORM OF THE FORMULA (LIE ALGEBRAS)

$$(Tf)(x_1, \ldots, x_n, x_{n+1})$$

$$= \sum_{j=1}^{n+1} (-1)^{n+1-j} [f(x_1, \ldots, \hat{x}_j, \ldots, x_{n+1}), x_j]$$

$$+ \sum_{j<k=2}^{n+1} (-1)^{j+k} f(x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_k, , \ldots, [x_j, x_k])$$

FIRST CASES

$\underline{n = 0}$

If f is any constant function from A to A, say, $f(x) = b$ for all x in A, where b is a fixed element of A, we have, consistent with the basic formula,

$$T_0(f)(x_1) = [b, x_1]$$

ASSOCIATIVE ALGEBRAS

$\underline{n = 1}$

If f is a linear map from A to A, then

$$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

$\underline{n = 2}$

If f is a bilinear map from $A \times A$ to A, then

$$T_2(f)(x_1, x_2, x_3) =$$

$$x_1 f(x_2, x_3) - f(x_1 x_2, x_3)$$

$$+ f(x_1, x_2 x_3) - f(x_1, x_2) x_3$$

LIE ALGEBRAS

$\underline{n = 1}$

If f is a linear map from A to A, then

$$T_1(f)(x_1, x_2) = -[f(x_2), x_1] + [f(x_1), x_2] - f([x_1, x_2])$$

$\underline{n = 2}$

If f is a skew-symmetric bilinear map from $A \times A$ to A, then

$$T_2(f)(x_1, x_2, x_3) =$$

$$[f(x_2, x_3), x_1] - [f(x_1, x_3), x_2] + [f(x_1, x_2), x_3]$$

$$-f(x_3, [x_1, x_2]) + f(x_2, [x_1, x_3]) - f(x_1, [x_2, x_3])$$

Kernel and Image of a linear transformation

- $G : X \to Y$

  Since X and Y are vector spaces, they are in particular, commutative groups.

- **Kernel** of G (also called **nullspace** of G) is
  $\ker G = \{x \in X : G(x) = 0\}$

  This is a subgroup of X

- **Image** of G is
  $\operatorname{im} G = \{G(x) : x \in X\}$

  This is a subgroup of Y

We now let $G = T_0, T_1, T_2$
(ASSOCIATIVE ALGEBRAS)

Kernel and Image of a linear transformation

- $G : X \to Y$

  Since X and Y are vector spaces, they are in particular, commutative groups.

- **Kernel** of G (also called **nullspace** of G) is
  $\ker G = \{x \in X : G(x) = 0\}$

  This is a subgroup of X

- **Image** of G is
  $\mathrm{im}\, G = \{G(x) : x \in X\}$

  This is a subgroup of Y

We now let $G = T_0, T_1, T_2$
(LIE ALGEBRAS)

$\underline{G = T_0}$

(ASSOCIATIVE ALGEBRAS)

$X = A$ (the algebra)

$Y = L(A)$ (all linear transformations on A)

$$T_0(f)(x_1) = x_1 b - b x_1$$

$\ker T_0 = \{b \in A : xb - bx = 0$ for all $x \in A\}$ (center of A)

$\operatorname{im} T_0 =$ the set of all linear maps of A of the form $x \mapsto xb - bx$,

in other words, the set of all inner derivations of A

$\ker T_0$ is a subgroup of A

$\operatorname{im} T_0$ is a subgroup of $L(A)$

$\underline{G = T_0}$

(LIE ALGEBRAS)

$X = A$ (the algebra)

$Y = L(A)$ (all linear transformations on A)

$T_0(f)(x_1) = [b, x_1]$

$\ker T_0 = \{b \in A : [b, x] = 0 \text{ for all } x \in A\}$
(center of A)

$\operatorname{im} T_0 =$ the set of all linear maps of A of the form $x \mapsto [b, x]$,

in other words, the set of all inner derivations of A

$\ker T_0$ is a subgroup of A

$\operatorname{im} T_0$ is a subgroup of $L(A)$

$\underline{G = T_1}$

(ASSOCIATIVE ALGEBRAS)

$X = L(A)$ (linear transformations on A)

$Y = L^2(A)$ (bilinear transformations on $A \times A$)

$$T_1(f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2$$

$\ker T_1 = \{f \in L(A) : T_1 f(x_1, x_2) = 0$ for all $x_1, x_2 \in A\}$ = the set of all derivations of A

$\operatorname{im} T_1$ = the set of all bilinear maps of $A \times A$ of the form

$$(x_1, x_2) \mapsto x_1 f(x_2) - f(x_1 x_2) + f(x_1) x_2,$$

for some linear function $f \in L(A)$.

$\ker T_1$ is a subgroup of $L(A)$

$\operatorname{im} T_1$ is a subgroup of $L^2(A)$

$\underline{G = T_1}$

(LIE ALGEBRAS)

$X = L(A)$ (linear transformations on A)

$Y = L^2(A)$ (bilinear transformations on $A \times A$)

$$T_1(f)(x_1, x_2) = -[f(x_2), x_1] + [f(x_1), x_2] - f([x_1, x_2])$$

$\ker T_1 = \{f \in L(A) : T_1 f(x_1, x_2) = 0$ for all $x_1, x_2 \in A\}$ = the set of all derivations of A

$\operatorname{im} T_1$ = the set of all bilinear maps of $A \times A$ of the form

$$(x_1, x_2) \mapsto -[f(x_2), x_1] + [f(x_1), x_2] - f([x_1, x_2])$$

for some linear function $f \in L(A)$.

$\ker T_1$ is a subgroup of $L(A)$

$\operatorname{im} T_1$ is a subgroup of $L^2(A)$

ASSOCIATIVE AND LIE ALGEBRAS

$$L^0(A) \xrightarrow{T_0} L(A) \xrightarrow{T_1} L^2(A) \xrightarrow{T_2} L^3(A) \cdots$$

FACTS:

- $T_1 \circ T_0 = 0$
- $T_2 \circ T_1 = 0$
- $\cdots$
- $T_{n+1} \circ T_n = 0$
- $\cdots$

Therefore

$$\operatorname{im} T_n \subset \ker T_{n+1} \subset L^n(A)$$

and

$\operatorname{im} T_n$ is a subgroup of $\ker T_{n+1}$

The cohomology groups of A are defined as the quotient groups

$$H^n(A) = \frac{\ker T_n}{\operatorname{im} T_{n-1}}$$

$$(n = 1, 2, \ldots)$$

Thus

$$H^1(A) = \frac{\ker T_1}{\operatorname{im} T_0} = \frac{\text{derivations}}{\text{inner derivations}}$$

$$H^2(A) = \frac{\ker T_2}{\operatorname{im} T_1} = \frac{?}{?}$$

The theorem that every derivation of $M_n(\mathbf{R})$ is inner (that is, of the form δ_a for some $a \in M_n(\mathbf{R})$) can now be restated as:

"the cohomology group $H^1(M_n(\mathbf{R}))$ is the trivial one element group"

$\underline{G = T_2}$

(ASSOCIATIVE ALGEBRAS)

$X = L^2(A)$ (bilinear transformations on $A \times A$)

$Y = L^3(A)$ (trilinear transformations on $A \times A \times A$)

$$T_2(f)(x_1, x_2, x_3) = x_1 f(x_2, x_3) - f(x_1 x_2, x_3) + f(x_1, x_2 x_3) - f(x_1, x_2) x_3$$

$\ker T_2 = \{f \in L(A) : T_2 f(x_1, x_2, x_3) = 0 \text{ for all } x_1, x_2, x_3 \in A\} =?$

$\operatorname{im} T_2 =$ the set of all trilinear maps h of $A \times A \times A$ of the form[§]

$$h(x_1, x_2, x_3) = x_1 f(x_2, x_3) - f(x_1 x_2, x_3) + f(x_1, x_2 x_3) - f(x_1, x_2) x_3,$$

for some bilinear function $f \in L^2(A)$.

$\ker T_2$ is a subgroup of $L^2(A)$

$\operatorname{im} T_2$ is a subgroup of $L^3(A)$

[§]we do not use $\operatorname{im} T_2$ in what follows

$\underline{G = T_2}$

(LIE ALGEBRAS)

$X = L_s^2(A)$ (skew symmetric bilinear transformations on $A \times A$)

$Y = L_s^3(A)$ (skew symmetric trilinear transformations on $A \times A \times A$)

$$T_2(f)(x_1, x_2, x_3) =$$

$$[f(x_2, x_3), x_1] - [f(x_1, x_3), x_2] + [f(x_1, x_2), x_3]$$

$$-f(x_3, [x_1, x_2]) + f(x_2, [x_1, x_3]) - f(x_1, [x_2, x_3])$$

$\ker T_2 = \{f \in L(A) : T_2 f(x_1, x_2, x_3) = 0$ for all $x_1, x_2, x_3 \in A\} =$?

$\ker T_2$ is a subgroup of $L^2(A)$

$\operatorname{im} T_2$ is a subgroup¶ of $L^3(A)$

¶we do not use $\operatorname{im} T_2$ in what follows

THIS PAGE PURPOSELY LEFT BLANK

Cohomology groups were defined in various contexts as follows

- associative algebras (1945)
- Lie algebras (1952)
- Lie triple systems (1961,2002)
- Jordan algebras (1971)
- associative triple systems (1976)
- Jordan triple systems (1982)

GRADUS AD PARNASSUM (COHOMOLOGY)

1. Verify that there is a one to one correspondence between partitions of a set X and equivalence relations on that set.
   Precisely, show that
   - If $X = \cup X_i$ is a partition of X, then $R := \{(x, y) \times X : x, y \in X_i$ for some $i\}$ is an equivalence relation whose equivalence classes are the subsets X_i.
   - If R is an equivalence relation on X with equivalence classes X_i, then $X = \cup X_i$ is a partition of X.

2. Verify that $T_{n+1} \circ T_n = 0$ for $n = 0, 1, 2$. Then prove it for all $n \geq 3$.

3. Show that if $f : G_1 \to G_2$ is a homomorphism of groups, then $G_1/\ker f$ is isomorphic to $f(G_1)$
   Hint: Show that the map $[x] \mapsto f(x)$ is an isomorphism of $G_1/\ker f$ onto $f(G_1)$

4. Show that if $h : A_1 \to A_2$ is a homomorphism of algebras, then $A_1/\ker h$ is isomorphic to $h(A_1)$
   Hint: Show that the map $[x] \mapsto h(x)$ is an isomorphism of $A_1/\ker h$ onto $h(A_1)$

5. Show that the algebra M_h in Example 2 is associative.
   Hint: You use the fact that A is associative AND the fact that, since $h \in \ker T_2$, $h(a,b)c + h(ab.c) = ah(b,c) + h(a,bc)$

6. Show that equivalence of extensions is actually an equivalence relation.
   Hint:
   - reflexive: $\psi : M \to M$ is the identity map
   - symmetric: replace $\psi : M \to M'$ by its inverse $\psi^{-1} : M' \to M$
   - transitive: given $\psi : M \to M'$ and $\psi' : M' \to M''$ let $\psi'' = \psi' \circ \psi : M \to M''$

7. Show that in example 2, if h_1 and h_2 are equivalent bilinear maps, that is, $h_1 - h_2 = T_1 f$ for some linear map f, then M_{h_1} and M_{h_2} are equivalent extensions of $\{0\} \times A$ by A. **Hint:** $\psi : M_{h_1} \to M_{h_2}$ is defined by
   $$\psi(a,x) = (a, x + f(a))$$