Maps which preserve equality of distance. Must they be linear? (Part 2)

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Abstract

I will talk about Tingley's problem for finite dimensional JC*-triples, basically rectangular matrices. Part 1 was about Tingley's problem for finite dimensional C*-algebras, basically square matrices. The main ingredient for the latter was the use of unitary matrices. In rectangular matrices there are no unitaries, so a new idea is needed. It was provided by Polo and Peralta.

Reference: Polo-Peralta

Francisco J. Fernandez-Polo and Antonio M. Peralta
Low rank compact operators and Tingley's problem (preprint 2016)

Notation

If X is a Banach space with norm $\|\cdot\|$, its unit ball and unit sphere are

$$B = B(X) = \{x \in X : \|x\| \leq 1\}$$

$$S = S(X) = \{x \in X : \|x\| = 1\}$$

Mazur-Ulam 1932

If $f; X \mapsto X'$ is a surjective isometry (not assumed linear), then f is linear (or more precisely, affine)

Mankiewicz 1972

If $f : B \rightarrow B'$ is a surjective isometry, then f extends to a linear (or affine) surjective isometry from X to X'.

Example 1

An isometry that is not linear or affine (Hint: it is not onto):
$X = \mathbb{R}, X' = \mathbb{R}^2, \|(x,y)\| = \max\{|x|,|y|\}$, $f(x) = (x,x)$ if $x \geq 0$, $f(x) = (x,-x)$ if $x < 0$.

Question (Tingley's problem)

If $f : S \to S'$ is a surjective isometry, is f the restriction to S of a linear (or affine) transformation? Not known at this time even for dimension 2.

Theorem (Tingley 1987)

Suppose that S and S' are the unit spheres of finite dimensional Banach spaces X and X'. If $f : S \to S'$ is a surjective isometry, then $f(-x) = -f(x)$ for all $x \in S$

Example 2

$X = \mathbb{R}$, $\|x\| = |x|$, $S = \{-1, 1\}$
There are four functions $f : S \to S$, two of which are surjective.

To make progress, give X some more structure.

The main result of Part 1

Theorem (Tanaka 2017)

Let A_1 and A_2 be finite dimensional C*-algebras. Suppose that $f : S(A_1) \to S(A_2)$ is a surjective isometry. Then there is a real **linear** surjective isometry $\phi : A_1 \to A_2$ such that $\phi(a) = f(a)$ for every $a \in S(A_1)$

A finite dimensional C*-algebra A is a finite direct sum of full matrix algebras:

$$A \sim M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots M_{n_k}(\mathbb{C})$$

$(\dim A = n_1^2 + n_2^2 + \cdots n_k^2)$

One of the main tools for Part 1

Theorem (Hatori-Molnar 2014) (isometries of unitaries)

Let $U(n)$ be the set of unitary $n \times n$ matrices. The map $\phi : U(n) \to U(n)$ is a surjective isometry if and only if there is a unitary $w \in U(n)$ such that either

$$\phi(a) = \phi(1)waw^* \text{ for all } a \in U(n)$$

or

$$\phi(a) = \phi(1)wa^t w^* \text{ for all } a \in U(n).$$

Some lemmas from part 1

(convex subsets of unit sphere) **Tanaka 2016, 2017**

Lemma 1

Let X be a Banach space. Suppose that C is a maximal convex subset of the unit sphere $S(X)$ of X. Then C is a norm exposed face of $B(X)$.

Convex set, extreme point, face, exposed point, exposed face

Lemma 2

Let X and Y be Banach spaces, and let $T : S(X) \to S(Y)$ be a surjective isometry. Then C is a maximal convex subset of $S(X)$ if and only if $T(C)$ is that of $S(Y)$. (So by Lemma 1, faces are mapped into faces.)

Lemma 3

Let A_1 and A_2 be finite dimensional C*-algebras. Suppose that $T : S(A_1) \to S(A_2)$ is a surjective isometry. Then T is locally affine, that is, if $a, b, ta + (1-t)b \in S(A_1)$ for some $t \in (0,1)$, then $sa + (1-s)b \in S(A_1)$ for all $s \in [0,1]$, and

$$T(sa + (1-s)b) = sT(a) + (1-s)T(b), \quad s \in [0,1].$$

faces in the unit ball

Theorem

Every closed face of $B(M_n(\mathbb{C}))$ is associated with a unique partial isometry $v \in M_n(\mathbb{C})$ such that

$$F = v + (1 - vv^*)B(M_n(\mathbb{C}))(1 - v^*v) = \{a \in M_n(\mathbb{C}) : av^* = vv^*\}.$$

More generally

Every closed face of $B(M_{m,n}(\mathbb{C}))$ is associated with a unique partial isometry $v \in M_{m,n}(\mathbb{C})$ such that

$$F = v + (1 - vv^*)B(M_{m,n}(\mathbb{C}))(1 - v^*v) = \{a \in M_n(\mathbb{C}) : av^* = vv^*\}.$$

The same is true if $M_{m,n}(\mathbb{C})$ is replaced by any Cartan factor.

Part 2 begins here

Recall

A finite dimensional C*-algebra A is a finite direct sum of full matrix algebras:

$$A \sim M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots M_{n_k}(\mathbb{C})$$

$(\dim A = n_1^2 + n_2^2 + \cdots n_k^2)$

A finite dimensional JC*-triple is a finite direct sum of **Cartan factors**:

$$A \sim C_1 \oplus C_2 \cdots \oplus C_k$$

Cartan factors

Type 1: $M_{m,n}(\mathbb{C})$, dimension mn, rank $\min\{m, n\}$
Type 2: $A_n(\mathbb{C})$ (anti-symmetric), dimension $(n-1)n/2)$, rank $[n/2]$
Type 3: $S_n(\mathbb{C})$ (symmetric), dimension $(n+1)n/2$, rank n
Type 4: Sp_n (spin factors), dimension n, rank 2
Type 5: $M_{1,2}(\mathcal{O})$, dimension 16, rank 2 ($\mathcal{O}$ = Octonions)
Type 6: $M_3(\mathcal{O})_h$, dimension 27, rank 3 ($\mathcal{O}$ = Octonions)

Theorem (Polo-Peralta Theorem 4.7)

Let A_1 and A_2 be finite dimensional JC*-triples. Suppose that $f : S(A_1) \to S(A_2)$ is a surjective isometry. Then there is a real **linear** surjective isometry $\phi : A_1 \to A_2$ such that $\phi(a) = f(a)$ for every $a \in S(A_1)$

This is just one step in the proof of

Theorem (Polo-Peralta Theorem 2.5)

Let $f : S(A) \to S(B)$ be a surjective isometry between the unit spheres of two weakly compact JB*-triples. Then there is a surjective real linear isometry $T : A \to B$ satisfying $T(x) = f(x)$ for every $x \in S(A)$.

For Theorem 4.7, it suffices to assume that A_1 and A_2 are Cartan factors, now called C and C'

We first dispose of the case when the rank of C is 1 (see the next page)

For C with rank ≥ 2, the proof is by induction on the dimension of C.

Rank 1 case

Guanggui Ding: Science in China 45 2002 pp 479-483

"The 1-Lipschitz mapping between the unit spheres of two Hilbert spaces can be extended to a real linear isometry of the whole space"

Theorem

Let E and F be inner-product spaces and V a mapping from the unit sphere $S_1(E)$ into $S_1(F)$. If $-V(S_1(E)) \subset V(S_1(E))$ and

$$\|V(x_1) - V(x_2)\| \leq \|x_1 - x_2\|, \text{ for all } x_1, x_2 \in S_1(E),$$

then V can be extended to a real-linear isometric mapping of E into F, Let $f : S(C) \to S(C')$ be a surjective isometry where C and C' are Cartan factors.

We need to prove that there is a real **linear** surjective isometry $\phi : C \to C'$ such that $\phi(a) = f(a)$ for every $a \in S(C)$

The statement is true if $\dim C = 1$

We assume that the statement is true for all Cartan factors of dimension $\leq n$ and suppose C has dimension $n + 1$ and rank ≥ 2.

Polo-Peralta Lemma 3.6

If e is a minimal partial isometry in C, then $f(e)$ is a minimal partial isometry in C', and either $f(ie) = if(e)$ or $f(ie) = -if(e)$

Polo-Peralta Corollary 3.13

- If $f(ie) = if(e)$ then $f(iu) = if(u)$ for **all** minimal partial isometries u in C
- If $f(ie) = -if(e)$ then $f(iu) = -if(u)$ for **all** minimal partial isometries u in C.

Idea for Lemma 3.6

If e is a minimal partial isometry in C, then

$$F = e + (1 - ee^*)B(C)(1 - e^*e)$$

is a face in $B(C)$. Since f maps faces to faces,

$$f(F) = v + (1 - vv^*)B(C')(1 - v^*v)$$

for some partial isometry $v \in C'$. It follows (details omitted) that v is minimal, $f(e) = v$, and $f(ie) = \pm if(e)$.

Corollary 3.13

details omitted

Polo-Peralta Proposition 2.1(c)

There is a surjective real linear isometry $T_e : C_0(e) \to C_0(f(e))$ satisfying $T_e(x) = f(x)$ for all $x \in S(C_0(e))$.

Polo-Peralta Lemma 3.14

If e is a minimal partial isometry in C, then $f(S(C_1(e))) = S(C_1(f(e)))$

Since dim $C_1(e) \leq n$ and $C_1(e)$ is a Cartan factor, by the induction hypothesis, we have a surjective real-linear isometry $T_1 : C_1(e) \to C_1(f(e))$ satisfying $T_1(x) = f(x)$ for all $x \in S(C_1(e))$.

Define a real linear mapping $T : C \to C'$ by

$$Tx = T(\lambda e + x_1 + x_0) = \lambda f(e) + T_1(x_1) + T_e(x_0),$$

for $x = \lambda e + x_1 + x_0 \in C_2(e) + C_1(e) + C_0(e)$

Polo-Peralta Theorem 4.5

$Tu = f(u)$ for every minimal partial isometry in C

Polo-Peralta Proposition 2.1(c), continued

$Tx = f(x)$ for every $x \in S(C)$

Q. E. D.