DERIVATIONS ON FINITE AND INFINITE DIMENSIONAL ALGEBRAS AND TRIPLE SYSTEMS

BERNARD RUSSO
University of California, Irvine

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I. Derivations on finite dimensional algebras

II. Derivations on operator algebras

III. Derivations on finite dimensional triple systems

IV. Derivations on Banach triple systems

I—DERIVATIONS ON FINITE DIMENSIONAL ALGEBRAS

Sophus Lie (1842–1899)

Marius Sophus Lie was a Norwegian mathematician. He largely created the theory of continuous symmetry, and applied it to the study of geometry and differential equations.

Pascual Jordan (1902–1980)

Pascual Jordan was a German theoretical and mathematical physicist who made significant contributions to quantum mechanics and quantum field theory.

LET $M_n(\mathbf{C})$ DENOTE THE ALGEBRA OF ALL n by n COMPLEX MATRICES, OR MORE GENERALLY, ANY FINITE DIMENSIONAL SEMISIMPLE ASSOCIATIVE ALGEBRA. .

DEFINITION 2

A DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO MATRIX MULTIPLICATION IS A LINEAR MAPPING δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(AB) = \delta(A)B + A\delta(B)$$

.

PROPOSITION 2

FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE

$$\delta_A(X) = AX - XA.$$

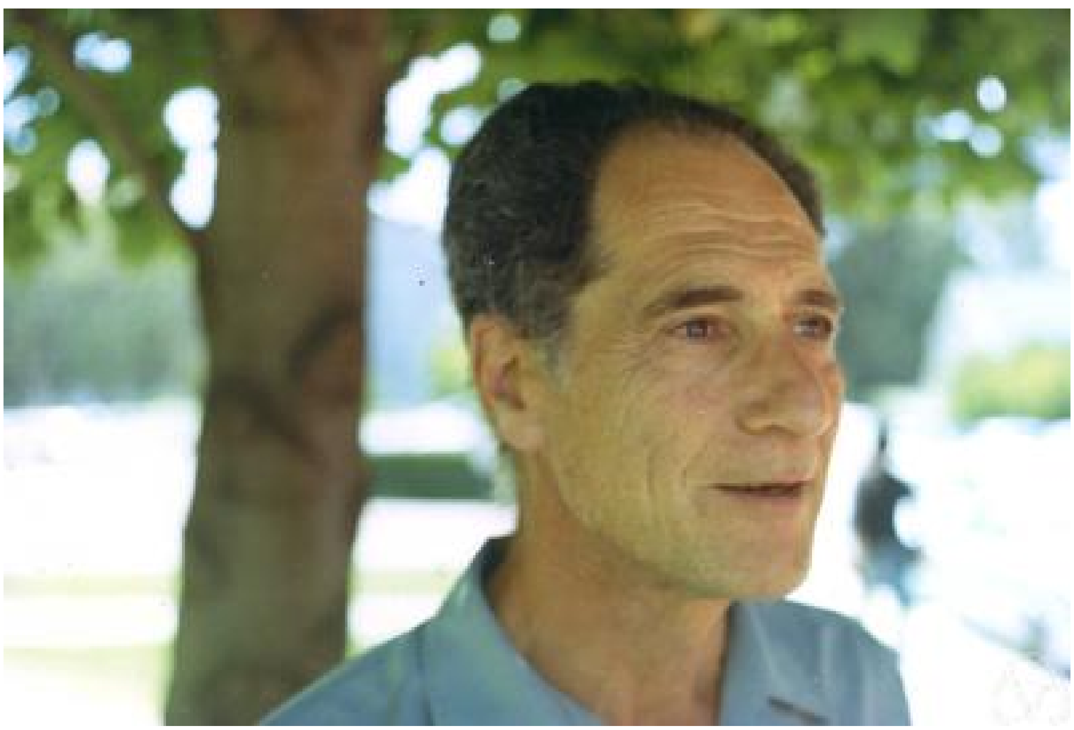
THEN δ_A IS A DERIVATION WITH RESPECT TO MATRIX MULTIPLICATION

THEOREM 2

(1933—Noether) (1937—Jacobson)
(1942—Hochschild) (Wedderburn)

EVERY DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO MATRIX MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{C})$.

Gerhard Hochschild (1915–2010)



(Photo 1968)

Gerhard Paul Hochschild was an American mathematician who worked on Lie groups, algebraic groups, homological algebra and algebraic number theory.

Joseph Henry Maclagan Wedderburn
(1882–1948)

Scottish mathematician, who taught at Princeton University for most of his career. A significant algebraist, he proved that a finite division algebra is a field, and part of the Artin–Wedderburn theorem on simple algebras. He also worked on group theory and matrix algebra.

Amalie Emmy Noether (1882–1935)

Amalie Emmy Noether was an influential German mathematician known for her groundbreaking contributions to abstract algebra and theoretical physics. Described as the most important woman in the history of mathematics, she revolutionized the theories of rings, fields, and algebras. In physics, Noether's theorem explains the fundamental connection between symmetry and conservation laws.

PROOF OF THEOREM 2
(Jacobson 1937)

If δ is a derivation, consider the two representations of $M_n(\mathbf{C})$

$$z \mapsto \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \text{ and } z \mapsto \begin{bmatrix} z & 0 \\ \delta(z) & z \end{bmatrix}$$

The first is a direct sum of two copies of the identity representation; but so is the second, since

$$\begin{bmatrix} 0 & 0 \\ \delta(z) & z \end{bmatrix} \text{ is equivalent to } \begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix}$$

$$\text{so } \begin{bmatrix} z & 0 \\ \delta(z) & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$$

Thus $az = za, \; bz = zb$

$\delta(z)a = cz - zc$ and $\delta(z)b = dz - zd.$

a and b are multiples of I and can't both be zero. QED

DEFINITION 3

A DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO BRACKET MULTIPLICATION

$$[X, Y] = XY - YX$$

IS A LINEAR MAPPING δ WHICH SATISFIES THE PRODUCT RULE

$$\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$$

.

PROPOSITION 3

FIX A MATRIX A in $M_n(\mathbf{C})$ AND DEFINE

$$\delta_A(X) = [A, X] = AX - XA.$$

THEN δ_A IS A DERIVATION WITH RESPECT TO BRACKET MULTIPLICATION

THEOREM 3
(1894 Cartan, 1942 Hochschild, Zassenhaus)
EVERY DERIVATION ON $M_n(\mathbf{C})^*$ WITH RESPECT TO BRACKET MULTIPLICATION IS OF THE FORM δ_A FOR SOME A IN $M_n(\mathbf{C})$.

Hans Zassenhaus (1912–1991)

Hans Julius Zassenhaus was a German mathematician, known for work in many parts of abstract algebra, and as a pioneer of computer algebra.

*not a semisimple Lie algebra: trace$(X)\,I$ is a derivation which is not inner

Elie Cartan 1869–1951

Elie Joseph Cartan was an influential French mathematician, who did fundamental work in the theory of Lie groups and their geometric applications. He also made significant contributions to mathematical physics, differential geometry, and group theory. He was the father of another influential mathematician, Henri Cartan.

PROOF OF THEOREM 3
(From Meyberg Notes 1972—Chapter 5)

An algebra L with multiplication $(x, y) \mapsto [x, y]$ is a Lie algebra if $[xx] = 0$ and

$$[[xy]z] + [[yz]x] + [[zx]y] = 0.$$

Left multiplication in a Lie algebra is denoted by ad(x): ad$(x)(y) = [x, y]$. An associative algebra A becomes a Lie algebra A^- under the product, $[xy] = xy - yx$.

The first axiom implies that $[xy] = -[yx]$ and the second (called the *Jacobi identity*) implies that $x \mapsto \mathrm{ad}x$ is a homomorphism of L into the Lie algebra $(\mathrm{End}\, L)^-$, that is, $\mathrm{ad}\,[xy] = [\mathrm{ad}\, x, \mathrm{ad}\, y]$.

Assuming that L is finite dimensional, the Killing form is defined by $\lambda(x, y) = \mathrm{tr}\,\mathrm{ad}(x)\mathrm{ad}(y)$.

CARTAN CRITERION

A finite dimensional Lie algebra L over a field of characteristic 0 is semisimple if and only if the Killing form is nondegenerate.

A linear map D is a derivation if $D \cdot \mathrm{ad}(x) = \mathrm{ad}(Dx) + \mathrm{ad}(x) \cdot D$. Each $\mathrm{ad}(x)$ is a derivation, called an inner derivation.

THEOREM OF E. CARTAN

If the finite dimensional Lie algebra L over a field of characteristic 0 is semisimple, then every derivation is inner.

PROOF

Let D be a derivation of L. Since $x \mapsto \mathrm{tr}\, D \cdot \mathrm{ad}\,(x)$ is a linear form, there exists $d \in L$ such that $\mathrm{tr}\, D \cdot \mathrm{ad}\,(x) = \lambda(d, x) = \mathrm{tr}\, \mathrm{ad}\,(d) \cdot \mathrm{ad}\,(x)$. Let E be the derivation $E = D - \mathrm{ad}\,(d)$ so that

$$\mathrm{tr}\, E \cdot \mathrm{ad}\,(x) = 0. \qquad (1)$$

Note next that

$E \cdot [\text{ad}(x), \text{ad}(y)] =$
$E \cdot \text{ad}(x) \cdot \text{ad}(y) - E \cdot \text{ad}(y) \cdot \text{ad}(x) =$
$(\text{ad}(x) \cdot E + [E, \text{ad}(x)]) \cdot \text{ad}(y) - E \cdot \text{ad}(y) \cdot \text{ad}(x)$

so that

$[E, \text{ad}(x)] \cdot \text{ad}(y)$
$= E \cdot [\text{ad}(x), \text{ad}(y)] - \text{ad}(x) \cdot E \cdot \text{ad}(y) + E \cdot \text{ad}(y) \cdot \text{ad}(x)$
$= E \cdot [\text{ad}(x), \text{ad}(y)] + [E \cdot \text{ad}(y), \text{ad}(x)]$

and

$$\text{tr}\,[E, \text{ad}(x)] \cdot \text{ad}(y) = \text{tr}\,E \cdot [\text{ad}(x), \text{ad}(y)].$$

However, since E is a derivation
$[E, \text{ad}(x)] \cdot \text{ad}(y)$
$= E \cdot \text{ad}(x) \cdot \text{ad}(y) - \text{ad}(x) \cdot E \cdot \text{ad}(y)$
$= (\text{ad}(Ex) + \text{ad}(x) \cdot E) \cdot \text{ad}(y) - \text{ad}(x) \cdot E \cdot \text{ad}(y)$
$= \text{ad}(Ex) \cdot \text{ad}(y).$

Thus

$$\begin{aligned} \lambda(Ex, y) &= \text{tr}\,\text{ad}(Ex) \cdot \text{ad}(y) \\ &= \text{tr}\,[E, \text{ad}(x)] \cdot \text{ad}(y) \\ &= \text{tr}\,E \cdot [\text{ad}(x), \text{ad}(y)] = 0 \text{ by (1)}. \end{aligned}$$

Since x and y are arbitrary, $E = 0$ and so $D - \text{ad}(d) = 0$. QED

DEFINITION 4

A DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO CIRCLE MULTIPLICATION

$$X \circ Y = (XY + YX)/2$$

IS A LINEAR MAPPING δ WHICH SATISFIES THE PRODUCT RULE

$$\delta(A \circ B) = \delta(A) \circ B + A \circ \delta(B)$$

PROPOSITION 4

FIX TWO MATRICES A, B in $M_n(\mathbf{C})$ AND DEFINE

$$\delta_{A,B}(X) = A \circ (B \circ X) - B \circ (A \circ X).$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO CIRCLE MULTIPLICATION

THEOREM 4

(1937 Jacobson)

EVERY DERIVATION ON $M_n(\mathbf{C})$ WITH RESPECT TO CIRCLE MULTIPLICATION IS A SUM OF DERIVATIONS OF THE FORM $\delta_{A,B}$ FOR SOME As and Bs IN $M_n(\mathbf{C})$.

Nathan Jacobson (1910–1999)

Nathan Jacobson was an American mathematician who was recognized as one of the leading algebraists of his generation, and he was also famous for writing more than a dozen standard monographs.

PROOF OF THEOREM 4 (Jacobson 1949)

First note that for any algebra, D is a derivation if and only if $[R_a, D] = R_{Da}$.

If you polarize the Jordan axiom $(a^2 b)a = a^2(ba)$, you get $[R_a, [R_b, R_c]] = R_{A(b,a,c)}$ where $A(b, a, c) = (ba)c - b(ac)$ is the "associator".

From the commutative law $ab = ba$, you get

$$A(b, a, c) = [R_b, R_c]a$$

and so $[R_b, R_c]$ is a derivation, sums of which are called **inner**, forming an ideal in the Lie algebra of all derivations.

The **Lie multiplication algebra** L of the Jordan algebra A is the Lie algebra generated by the multiplication operators R_a. It is given by

$$L = \{R_a + \sum_i [R_{b_i}, R_{c_i}] : a, b_i, c_i \in A\}$$

so that L is the sum of a Lie triple system and the <u>ideal</u> of inner derivations.

Now let D be a derivation of a semisimple finite dimensional unital Jordan algebra A. Then $\tilde{D} : X \mapsto [X, D]$ is a derivation of L.

It is well known to algebraists that $L = L' + C$ where L' (the derived algebra $[L, L]$) is semisimple and C is the center of L. Also $\tilde{D}$ maps L' into itself and C to zero.

By Theorem 3, $\tilde{D}$ is an inner derivation of L' and hence also of L, so there exists $U \in L$ such that $[X, D] = [X, U]$ for all $X \in L$ and in particular $[R_a, D] = [R_a, U]$.

Then $Da = R_{Da}1 = [R_a, D]1 = [R_a, U]1 = (R_a U - U R_a)1 = a \cdot U1 - Ua$ so that $D = R_{U1} - U \in L$.

Thus, $D = R_a + \sum [R_{b_i}, R_{c_i}]$ and so

$$0 = D1 = a + 0 = a \quad \text{QED}$$

II—DERIVATIONS ON OPERATOR ALGEBRAS

A BASIC QUESTION ON DERIVATIONS OF BANACH ALGEBRAS (INTO THEMSELVES)

- ARE ALL (CONTINUOUS) DERIVATIONS INNER? IF NOT, WHY NOT?

POSSIBLE CONTEXTS

(i) C*-ALGEBRAS
(Banach ⟨associative⟩ algebras)

(ii) JC*-ALGEBRAS
(Banach Jordan algebras)

(iii) JC*-TRIPLES
(Banach Jordan triples)

Could also consider:

(ii') <u>Banach Lie algebras</u>
(iii')<u>Banach Lie triple systems</u>
(i')<u>Banach associative triple systems</u>

(i) C*-ALGEBRAS

derivation: $D(ab) = a(Db) + (Da)b$

inner derivation: ad a(x) $= xa - ax$ (a fixed)

THEOREM (1966-Sakai, Kadison)

EVERY DERIVATION OF A C^*-ALGEBRA IS OF THE FORM $x \mapsto xa - ax$ FOR SOME a IN THE WEAK CLOSURE OF THE C^*-ALGEBRA

Soichiro Sakai (b. 1926 c.)

Richard Kadison (b. 1925)

Richard V. Kadison is an American mathematician known for his contributions to the study of operator algebras.

(ii) JC*-ALGEBRA

derivation: $D(a \circ b) = a \circ Db + Da \circ b$

inner derivation: $\sum_i [L(x_i)L(a_i) - L(a_i)L(x_i)]$

$(x_i, a_i \in A)$

$b \mapsto \sum_i [x_i \circ (a_i \circ b) - a_i \circ (x_i \circ b)]$

THEOREM (1980-Upmeier)

1. Purely exceptional JBW-algebras have the inner derivation property
2. Reversible JBW-algebras have the inner derivation property
3. $\oplus L^\infty(S_j, U_j)$ has the inner derivation property if and only if $\sup_j \dim U_j < \infty$, U_j spin factors.

Harald Upmeier (b. 1950)

(ii') LIE OPERATOR ALGEBRAS

C. Robert Miers, Lie derivations of von Neumann algebras. DukeMath. J. 40 (1973), 403–409.

If M is a von Neumann algebra, $[M, M]$ the Lie algebra linearly generated by $\{[X, Y] = XY - YX : X, Y \in M\}$ and $L : [M, M] \to M$ a Lie derivation, i.e., L is linear and $L[X, Y] = [LX, Y] + [X, LY]$, then L has an extension $D : M \to M$ that is a derivation of the associative algebra.

The proof involves matrix-like computations.

Using the Sakai-Kadison theorem, Miers shows that if $L : M \to M$ is a Lie derivation, then $L = D + \lambda$, where D is an associative derivation and λ is a linear map into the center of M vanishing on $[M, M]$.

THEOREM (JOHNSON 1996)

EVERY <u>CONTINUOUS</u> LIE DERIVATION OF A C^*-ALGEBRA A INTO A BANACH BIMODULE X (IN PARTICULAR, $X = A$) IS THE SUM OF AN ASSOCIATIVE DERIVATION AND A "TRIVIAL" DERIVATION

(TRIVIAL=ANY LINEAR MAP WHICH VANISHES ON COMMUTATORS AND MAPS INTO THE "CENTER" OF THE MODULE).

"It is also an interesting open question whether or not every Lie derivation on a semisimple Banach algebra to itself has this form."

(H.G. Dales)

NOTE: Johnson's 1996 paper does not quote Miers's 1973 paper, which it partially but significantly generalizes.

III—DERIVATIONS ON FINITE DIMENSIONAL TRIPLE SYSTEMS

DEFINITION 5

A DERIVATION ON $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION

IS A LINEAR MAPPING δ WHICH SATISFIES THE (TRIPLE) PRODUCT RULE

$$\delta(AB^*C) =$$
$$\delta(A)B^*C + A\delta(B)^*C + AB^*\delta(C)$$

PROPOSITION 5

FOR TWO MATRICES $A \in M_m(\mathbf{C}), B \in \mathrm{M}_n(\mathbf{C})$, WITH $A^* = -A, B^* = -B$,

DEFINE $\delta_{A,B}(X) =$

$$AX + XB$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION

THEOREM 5

EVERY DERIVATION ON $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE MATRIX MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

REMARKS

1. THESE RESULTS HOLD TRUE AND ARE OF INTEREST FOR THE CASE $m = n$.
2. THEOREM 5 IS A CONSEQUENCE OF THEOREM 7, BY CONSIDERING THE SYMMETRIZED TRIPLE MATRIX MULTIPLICATION: $AB^*C + CB^*A$.

TRIPLE BRACKET MULTIPLICATION

LET'S GO BACK FOR A MOMENT TO SQUARE MATRICES AND THE BRACKET MULTIPLICATION.

MOTIVATED BY THE LAST REMARK, WE DEFINE THE TRIPLE BRACKET MULTIPLICATION TO BE $[[X, Y], Z]$

DEFINITION 6

A <u>DERIVATION</u> ON $M_n(\mathbf{C})$ WITH <u>RESPECT TO TRIPLE BRACKET MULTIPLICATION</u>

IS A LINEAR MAPPING δ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$$\delta([[A, B], C]) =$$
$$[[\delta(A), B], C] + [[A, \delta(B)], C] + [[A, B], \delta(C)]$$

PROPOSITION 6

FIX TWO MATRICES A, B IN $M_n(\mathbf{C})$ AND DEFINE $\delta_{A,B}(X) = [[A, B], X]$ THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION.

THEOREM 6

EVERY DERIVATION OF $M_n(\mathbf{C})^\dagger$ WITH RESPECT TO TRIPLE BRACKET MULTIPLICATION IS A SUM OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

† not a semisimple Lie triple system, as in Theorem 3

PROOF OF THEOREM 6

(From Meyberg Notes 1972—Chapter 6)

Let F be a finite dimensional semisimple Lie triple system over a field of characteristic 0 and suppose that D is a derivation of F.

Let L be the Lie algebra $(\text{Inder}\, F) \oplus F$ with product

$$[(H_1, x_1), (H_2, x_2)] =$$

$$([H_1, H_2] + L(x_1, x_2), H_1 x_2 - H_2 x_1).$$

A derivation of L is defined by $\delta(H \oplus a) = [D, H] \oplus Da$.

We take as a leap of faith that F semisimple implies L semisimple (IT'S TRUE!). Thus there exists $U = H_1 \oplus a_1 \in L$ such that $\delta(X) = [U, X]$ for all $X \in L$.

Then $0 \oplus Da = \delta(0 \oplus a) = [H_1 + a_1, 0 \oplus a] = L(a_1, a) \oplus H_1 a$ so $L(a_1, a) = 0$ and $D = H_1 \in \text{Inder}\, F$. QED

TRIPLE CIRCLE MULTIPLICATION

LET'S RETURN TO RECTANGULAR MATRICES AND FORM THE TRIPLE CIRCLE MULTIPLICATION

$$(AB^*C + CB^*A)/2$$

For sanity's sake, let us write this as

$$\{A, B, C\} = (AB^*C + CB^*A)/2$$

DEFINITION 7

A DERIVATION ON $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION

IS A LINEAR MAPPING δ WHICH SATISFIES THE TRIPLE PRODUCT RULE

$$\delta(\{A,B,C\}) = \{\delta(A), B, C\} + \{A, \delta(B), C\} + \{B, A, \delta(C)\}$$

PROPOSITION 7

FIX TWO MATRICES A, B IN $M_{m,n}(\mathbf{C})$ AND DEFINE

$$\delta_{A,B}(X) = \{A, B, X\} - \{B, A, X\}$$

THEN $\delta_{A,B}$ IS A DERIVATION WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION.

THEOREM 7

EVERY DERIVATION OF $M_{m,n}(\mathbf{C})$ WITH RESPECT TO TRIPLE CIRCLE MULTIPLICATION IS A **SUM** OF DERIVATIONS OF THE FORM $\delta_{A,B}$.

PROOF OF THEOREM 7

((From Meyberg Notes 1972—Chapter 11)‡

Let V be a Jordan triple and let $L(V)$ be its TKK Lie algebra (**Tits-Kantor-Koecher**)

$L(V) = V \oplus V_0 \oplus V$ and the Lie product is given by $[(x,h,y),(u,k,v)] =$
$(hu - kx,\, [h,k] + x \,\square\, v - u \,\square\, y,\, k^\natural y - h^\natural v)$.

$V_0 = \text{span}\{V \,\square\, V\}$ is a Lie subalgebra of $L(V)$ and for $h = \sum_i a_i \,\square\, b_i \in V_0$, the map $h^\natural : V \to V$ is defined by

$$h^\natural = \sum_i b_i \,\square\, a_i.$$

We can show the correspondence of derivations $\delta : V \to V$ and $D : L(V) \to L(V)$ for Jordan triple V and its TKK Lie algebra $L(V)$.

Let $\theta : L(V) \to L(V)$ be the main involution
$\theta(x \oplus h \oplus y) = y \oplus -h^\natural \oplus x$

‡slightly simplified by Chu and Russo 2012

LEMMA 1

Let $\delta : V \to V$ be a derivation of a Jordan triple V, with TKK Lie algebra $(L(V), \theta)$. Then there is a derivation $D : L(V) \to L(V)$ satisfying

$$D(V) \subset V \quad \text{and} \quad D\theta = \theta D.$$

PROOF

Given $a, b \in V$, we define

$$\begin{aligned} D(a, 0, 0) &= (\delta a, 0, 0) \\ D(0, 0, b) &= (0, 0, \delta b) \\ D(0, a \square b, 0) &= (0,\ \delta a \square b + a \square \delta b,\ 0) \end{aligned}$$

and extend D linearly on $L(V)$. Then D is a derivation of $L(V)$ and evidently, $D(V) \subset V$.

It is readily seen that $D\theta = \theta D$, since

$$\begin{aligned} D\theta(0, a \square b, 0) &= D(0, -b \square a, 0) \\ &= (0,\ -\delta b \square a - b \square \delta a,\ 0) \\ &= \theta(0,\ \delta a \square b + a \square \delta b,\ 0) \\ &= \theta D(0, a \square b, 0). \text{QED} \end{aligned}$$

LEMMA 2

Let V be a Jordan triple with TKK Lie algebra $(L(V), \theta)$. Given a derivation $D : L(V) \to L(V)$ satisfying $D(V) \subset V$ and $D\theta = \theta D$, the restriction $D|_V : V \to V$ is a triple derivation.

THEOREM

Let V be a Jordan triple with TKK Lie algebra $(L(V), \theta)$. There is a one-one correspondence between the triple derivations of V and the Lie derivations $D : L(V) \to L(V)$ satisfying $D(V) \subset V$ and $D\theta = \theta D$.

LEMMA 3

Let V be a Jordan triple with TKK Lie algebra $(L(V), \theta)$. Let $D : L(V) \to L(V)$ be a Lie inner derivation such that $D(V) \subset V$. Then the restriction $D|_V$ is a triple inner derivation of V.

COROLLARY

Let δ be a derivation of a finite dimensional semisimple Jordan triple V. Then δ is a triple inner derivation of V.

PROOF

The TKK Lie algebra $L(V)$ is semisimple. Hence the result follows from the Lie result and Lemma 3

The proof of lemma 3 is instructive.

1. $D(x,k,y) = [(x,k,y),(a,h,b)]$ for some $(a,h,b) \in (V)$
2. $D(x,0,0) = [(x,0,0),(a,h,b)] = (-h(x), x \square b, 0)$
3. $\delta(x) = -h(x) = -\sum_i \alpha_i \square \beta_i(x)$
4. $D(0,0,y) = [(0,0,y),(a,h,b)] = (0, -a \square y, h^\natural(y))$
5. $\delta(x) = -h^\natural(x) = \sum_i \beta_i \square \alpha_i(x)$
6. $\delta(x) = \frac{1}{2}\sum_i(\beta_i \square \alpha_i - \alpha_i \square \beta_i)(x)$

QED

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

AN <u>TRIPLE SYSTEM</u> IS DEFINED TO BE A SET (ACTUALLY A VECTOR SPACE) WITH ONE BINARY OPERATION, CALLED <u>ADDITION</u> AND ONE TERNARY OPERATION CALLED <u>TRIPLE MULTIPLICATION</u>

ADDITION IS DENOTED BY

$$a + b$$

AND IS REQUIRED TO BE COMMUTATIVE AND ASSOCIATIVE

$$a + b = b + a, \quad (a + b) + c = a + (b + c)$$

TRIPLE MULTIPLICATION IS DENOTED (TEMPORARILY) BY

$$abc$$

AND IS REQUIRED TO BE LINEAR IN EACH VARIABLE

$$(a + b)cd = acd + bcd$$
$$a(b + c)d = abd + acd$$
$$ab(c + d) = abc + abd$$

SIMPLE, BUT SOMETIMES IMPORTANT, EXAMPLES OF TRIPLE SYSTEMS CAN BE FORMED FROM ANY ALGEBRA

IF ab DENOTES THE ALGEBRA PRODUCT, JUST DEFINE A TRIPLE MULTIPLICATION TO BE $(ab)c$ (OR $a(bc)$)

LET'S SEE HOW THIS WORKS IN THE ALGEBRAS WE INTRODUCED EARLIER

1. $(M_n(\mathbf{C}), \times)$; abc OR ab^*c

OK:associative triple systems.

But $\{abc\} = abc + cba$ **is better: you get a Jordan triple system**

2. $(M_n(\mathbf{C}), [,])$, $[[a,b],c]$ **OK: Lie triple system**

3. $(M_n(\mathbf{C}), \circ)$; $abc = (a \circ b) \circ c$ **NO GO!**

$\{abc\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$ **is better: you get a Jordan triple system again**

A TRIPLE SYSTEM IS SAID TO BE ASSOCIATIVE (RESP. COMMUTATIVE) IF THE **MULTIPLICATION** IS ASSOCIATIVE (RESP. COMMUTATIVE)

(RECALL THAT ADDITION IS ALWAYS COMMUTATIVE AND ASSOCIATIVE)

IN THE TRIPLE CONTEXT THIS MEANS THE FOLLOWING

ASSOCIATIVE

$$ab(cde) = (abc)de = a(bcd)e$$

OR $ab(cde) = (abc)de = a(dcb)e$

COMMUTATIVE: $abc = cba$

AXIOMATIC APPROACH FOR TRIPLE SYSTEMS

THE AXIOM WHICH CHARACTERIZES TRIPLE MATRIX MULTIPLICATION IS

$$(abc)de = ab(cde) = a(dcb)e$$

THESE ARE CALLED

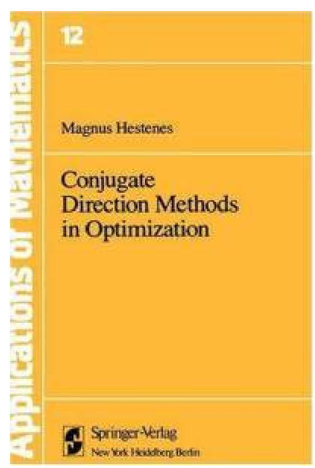
ASSOCIATIVE TRIPLE SYSTEMS

or

HESTENES ALGEBRAS

Magnus Hestenes (1906–1991)

Magnus Rudolph Hestenes was an American mathematician. Together with Cornelius Lanczos and Eduard Stiefel, he invented the conjugate gradient method.



THE AXIOMS WHICH CHARACTERIZE TRIPLE BRACKET MULTIPLICATION ARE

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

THESE ARE CALLED
LIE TRIPLE SYSTEMS

(NATHAN JACOBSON, MAX KOECHER)

Max Koecher (1924–1990)

Max Koecher was a German mathematician. His main research area was the theory of Jordan algebras, where he introduced the KantorKoecherTits construction.

Nathan Jacobson (1910–1999)

THE AXIOMS WHICH CHARACTERIZE TRIPLE CIRCLE MULTIPLICATION ARE

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

THESE ARE CALLED

JORDAN TRIPLE SYSTEMS

Kurt Meyberg

Ottmar Loos + Erhard Neher

Table 4

TRIPLE SYSTEMS

associative triple systems

$$(abc)de = ab(cde) = a(dcb)e$$

Lie triple systems

$$aab = 0$$

$$abc + bca + cab = 0$$

$$de(abc) = (dea)bc + a(deb)c + ab(dec)$$

Jordan triple systems

$$abc = cba$$

$$de(abc) = (dea)bc - a(edb)c + ab(dec)$$

IV—DERIVATIONS ON BANACH TRIPLES

(iii) JC*-TRIPLE

derivation:

$$D\{a,b,c\} = \{Da.b,c\} + \{a,Db,c\} + \{a,b,Dc\}$$

$$\{x,y,z\} = (xy^*z + zy^*x)/2$$

inner derivation: $\sum_i [L(x_i,a_i) - L(a_i,x_i)]$

$$(x_i, a_i \in A)$$

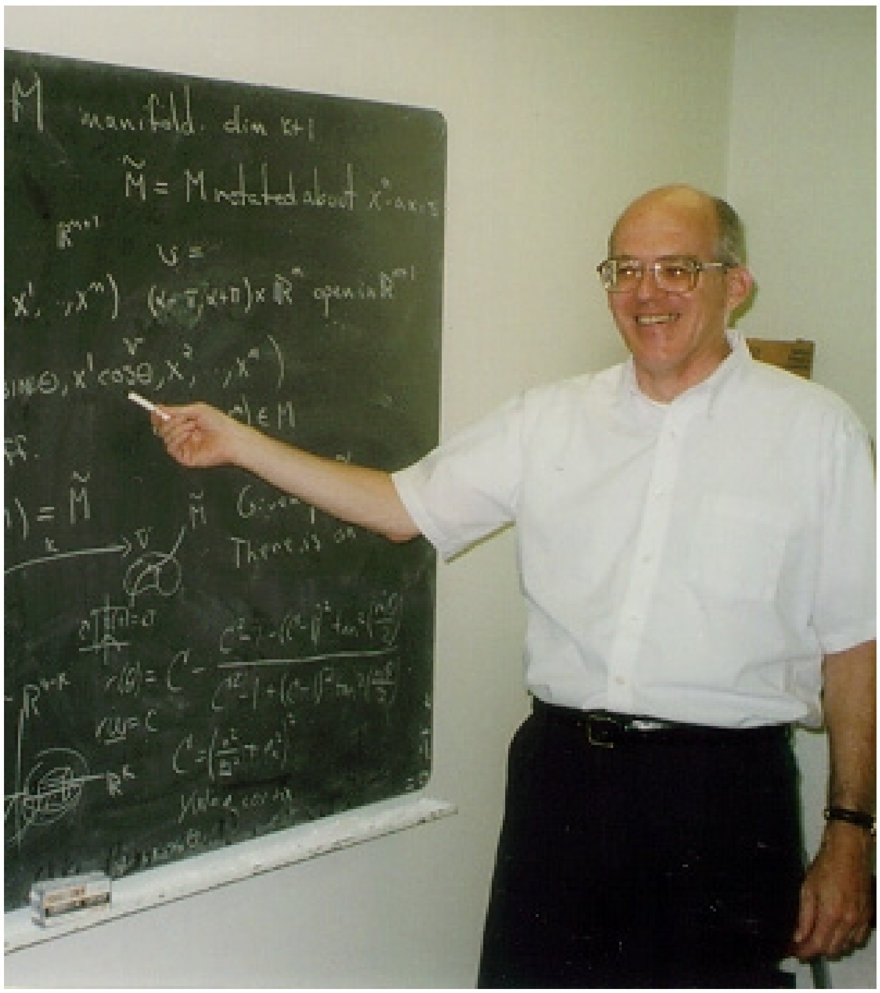
$$b \mapsto \sum_i [\{x_i,a_i,b\} - \{a_i,x_i,b\}]$$

INNER DERIVATION RESULTS

- HO-MARTINEZ-PERALTA-RUSSO 2002 (JBW^*-triples)

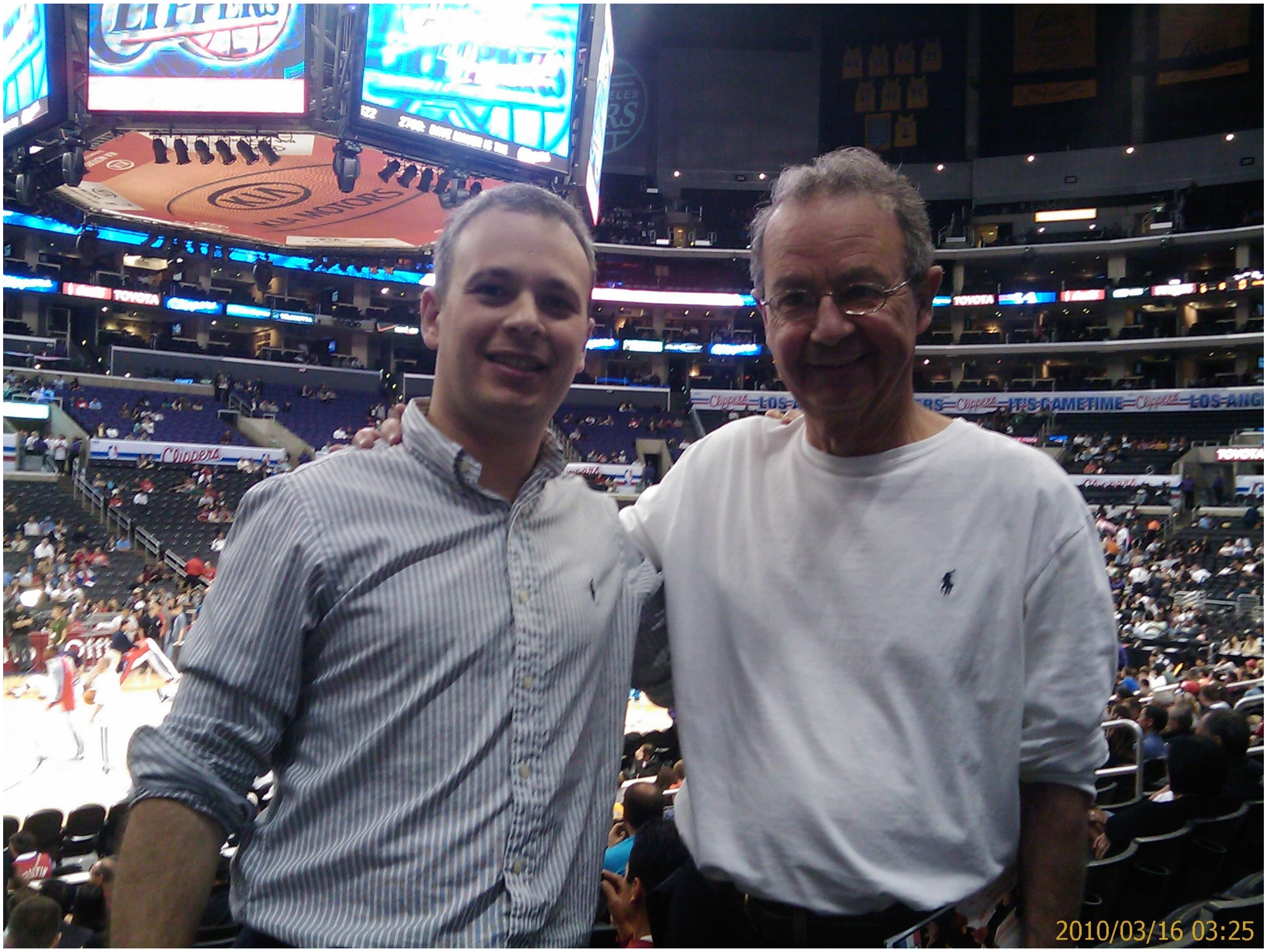
KUDOS TO:

Lawrence A. Harris (PhD 1969)



1974 (infinite dimensional holomorphy)

1981 (spectral and ideal theory)



Antonio Peralta (b. 1974)
Bernard Russo (b. 1939)

GO LAKERS!

THEOREM 2002
(Ho-Martinez-Peralta-Russo)
CARTAN FACTORS OF TYPE $I_{n,n}$, II (even or ∞), and III HAVE THE INNER DERIVATION PROPERTY

THEOREM 2002
(Ho-Martinez-Peralta-Russo)
INFINITE DIMENSIONAL CARTAN FACTORS OF TYPE $I_{m,n}, m \neq n$, and IV DO NOT HAVE THE INNER DERIVATION PROPERTY.

(iii') LIE OPERATOR TRIPLE SYSTEMS

C. Robert Miers, Lie triple derivations of von Neumann algebras. Proc. Amer. Math. Soc. 71 (1978), no. 1, 57–61.

Authors summary: A Lie triple derivation of an associative algebra M is a linear map $L : M \to M$ such that

$$L[[X,Y],Z] = [[L(X),Y],Z] +$$

$$[[X,L(Y)],Z] + [[X,Y],L(Z)]$$

for all $X, Y, Z \in M$.

We show that if M is a von Neumann algebra with no central Abelian summands then there exists an operator $A \in M$ such that $L(X) = [A,X] + \lambda(X)$ where $\lambda : M \to Z_M$ is a linear map which annihilates brackets of operators in M.

(i') ASSOCIATIVE OPERATOR TRIPLE SYSTEMS

Borut Zalar, On the structure of automorphism and derivation pairs of B*-triple systems. Topics in Operator Theory, operator algebras and applications (Timisoara,1994),265-271, Rom. Acad.,Bucharest, 1995

Let $W \subset B(H,K)$ be a TRO which contains all the compact operators. If D is a derivation of W with respect to the associative triple product ab^*c then there exist $a = -a^* \in B(K)$ and $b = -b^* \in B(H)$ such that $Dx = ax + xb$.

Extended to $B(X,Y)$ (X,Y Banach spaces) in

Maria Victoria Velasco and Armando R. Villena; Derivations on Banach pairs. Rocky Mountain J. Math 28 1998 1153–1187.