# TERNARY WEAKLY AMENABLE C*-ALGEBRAS

BERNARD RUSSO
University of California, Irvine

Report on joint work with

Antonio M. Peralta
Universidad de Granada, Spain

GPOTS XXXI
ARIZONA STATE UNIVERSITY

MAY 18, 2011

# TWO BASIC QUESTIONS CONCERNING DERIVATIONS ON BANACH ALGEBRAS

$A \to A$ and $A \to M$ (MODULE)

1. AUTOMATIC CONTINUITY?

2. INNER?

## CONTEXTS

I. C*-ALGEBRAS
(associative Banach algebras)
$ab \quad a^*$

II. JC*-ALGEBRAS
(Jordan Banach algebras)
$a^2 \quad a^*$

III.JC*-TRIPLES
(Banach Jordan triples)
$aa^*a$

# I. C*-ALGEBRAS

derivation: $D(ab) = a \cdot Db + Da \cdot b$

inner derivation: ad x$(a) = x \cdot a - a \cdot x$ $(x \in M)$

1. AUTOMATIC CONTINUITY RESULTS

   KAPLANSKY 1949: $C(X)$

   SAKAI 1960:

   RINGROSE 1972: (module)

2. INNER DERIVATION RESULTS

   SAKAI, KADISON 1966

   CONNES 1976 (module)

   HAAGERUP 1983 (module)

**THEOREM (Sakai 1960)**
Every derivation from a C*-algebra into itself is continuous.

**THEOREM (Ringrose 1972)**
Every derivation from a C*-algebra into a Banach $A$-**bimodule** is continuous.

**THEOREM (1966-Sakai, Kadison)**
EVERY DERIVATION OF A $C^*$-ALGEBRA IS OF THE FORM $x \mapsto ax - xa$ FOR SOME $a$ IN THE WEAK CLOSURE OF THE $C^*$-ALGEBRA

**POP QUIZ**: WHO PROVED THIS FOR $M_n(C)$?

**THEOREM (1976-Connes)**
EVERY AMENABLE $C^*$-ALGEBRA IS NUCLEAR.

**THEOREM (1983-Haagerup)**
EVERY NUCLEAR $C^*$-ALGEBRA IS AMENABLE.

**THEOREM (1983-Haagerup)**
EVERY $C^*$-ALGEBRA IS WEAKLY AMENABLE.

## DIGRESSION
## A BRIDGE TO JORDAN ALGEBRAS

A *Jordan derivation* from a Banach algebra $A$ into a Banach $A$-module is a linear map $D$ satisfying $D(a^2) = aD(a) + D(a)a$, $(a \in A)$, or equivalently,

$$D(ab + ba) = aD(b) + D(b)a + D(a)b + bD(a),$$
$$(a, b \in A).$$

Sinclair proved in 1970 that a bounded Jordan derivation from a semisimple Banach algebra to itself is a derivation, although this result fails for derivations of semisimple Banach algebras into a Banach bi-module.

Nevertheless, a celebrated result of B.E. Johnson in 1996 states that every bounded Jordan derivation from a C*-algebra $A$ to a Banach $A$-bimodule is an associative derivation.

In view of the intense interest in automatic continuity problems in the past half century, it is therefore somewhat surprising that the following problem has remained open for fifteen years.

**PROBLEM**

Is every Jordan derivation from a C$^*$-algebra $A$ to a Banach $A$-bimodule automatically continuous (and hence a derivation, by Johnson's theorem)?

In 2004, J. Alaminos, M. Brešar and A.R. Villena gave a positive answer to the above problem for some classes of C$^*$-algebras including the class of abelian C$^*$-algebras

Combining a theorem of Cuntz from 1976 with the theorem just quoted yields

**THEOREM**
**Every Jordan derivation from a C$^*$-algebra $A$ to a Banach $A$-module is continuous.**

In the same way, using the solution in 1996 by Hejazian-Niknam in the commutative case we have

**THEOREM**
**Every Jordan derivation from a C$^*$-algebra $A$ to a *Jordan* Banach $A$-module is continuous.**

(Jordan module will be defined below)

These two results will also be among the consequences of our results on automatic continuity of derivations into Jordan triple modules.

**(END OF DIGRESSION)**

## II. JC*-ALGEBRA

derivation: $D(a \circ b) = a \circ Db + Da \circ b$

inner derivation: $\sum_i [L(x_i)L(a_i) - L(a_i)L(x_i)]$

$$(x_i \in M, a_i \in A)$$

$$b \mapsto \sum_i [x_i \circ (a_i \circ b) - a_i \circ (x_i \circ b)]$$

1. AUTOMATIC CONTINUITY RESULTS

   UPMEIER 1980

   HEJAZIAN-NIKNAM 1996 (module)

   ALAMINOS-BRESAR-VILLENA 2004 (module)

2. INNER DERIVATION RESULTS

   JACOBSON 1951 (module)

   UPMEIER 1980

**THEOREM (1951-Jacobson)**
EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN ALGEBRA INTO A (JORDAN) **MODULE** IS INNER
(Lie algebras, Lie triple systems)

**THEOREM (1980-Upmeier)**
EVERY DERIVATION OF A REVERSIBLE JC*-ALGEBRA EXTENDS TO A DERIVATION OF ITS ENVELOPING C*-ALGEBRA. (IMPLIES SINCLAIR)

**THEOREM (1980-Upmeier)**

1. Purely exceptional JBW-algebras have the inner derivation property
2. Reversible JBW-algebras have the inner derivation property
3. $\oplus L^{\infty}(S_j, U_j)$ has the inner derivation property if and only if $\sup_j \dim U_j < \infty$, $U_j$ spin factors.

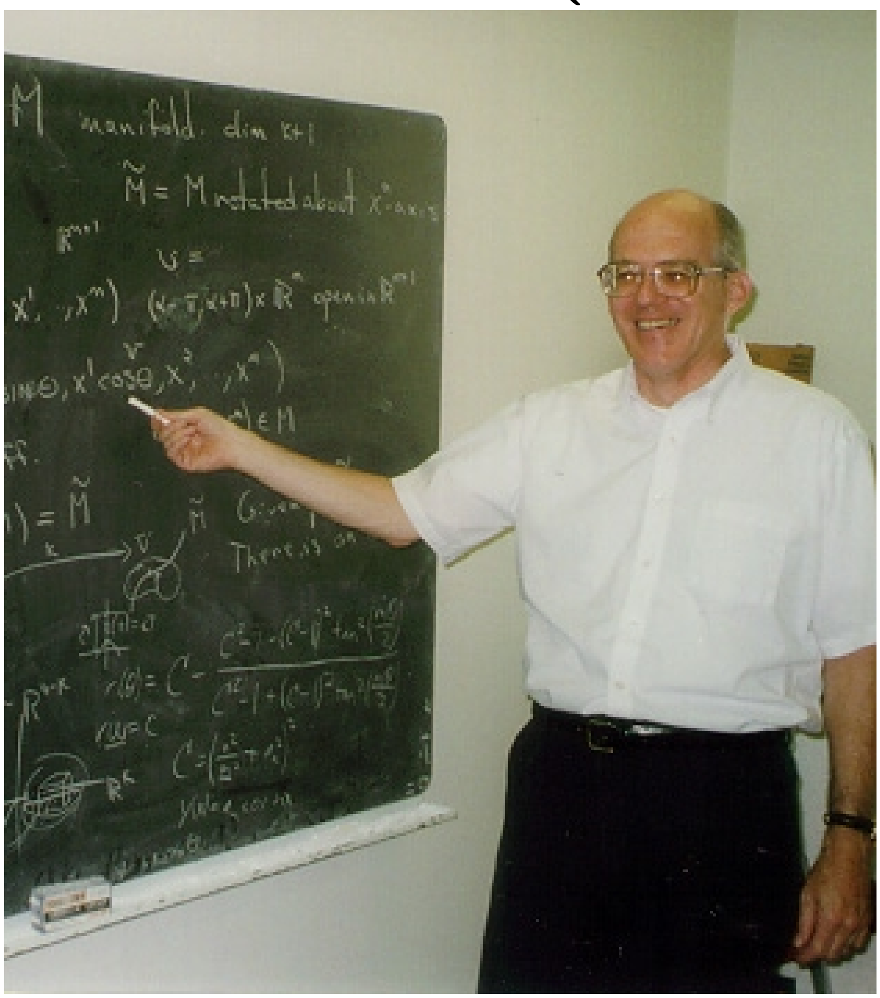
**Nathan Jacobson (1910-1999)**

**Harald Upmeier (b. 1950)**

III. JC*-TRIPLE

KUDOS TO:

**Lawrence A. Harris (PhD 1969)**



1974 (infinite dimensional holomorphy)

1981 (spectral and ideal theory)

$$\{x, y, z\} = (xy^*z + zy^*x)/2$$

derivation:

$$D\{a,b,c\} = \{Da.b,c\} + \{a,Db,c\} + \{a,b,Dc\}$$

inner derivation: $\sum_i [L(x_i,a_i) - L(a_i,x_i)]$

$$(x_i \in M, a_i \in A)$$

$$b \mapsto \sum_i [\{x_i,a_i,b\} - \{a_i,x_i,b\}]$$

1. AUTOMATIC CONTINUITY RESULTS

   BARTON-FRIEDMAN 1990

   (**NEW**) PERALTA-RUSSO 2010 (module)

2. INNER DERIVATION RESULTS

   HO-MARTINEZ-PERALTA-RUSSO 2002

   MEYBERG 1972

   (**NEW**) PERALTA-RUSSO 2011 (module)
   (weak amenability)

**AUTOMATIC CONTINUITY RESULTS**

**THEOREM (1990 Barton-Friedman)**
EVERY DERIVATION OF A JB*-TRIPLE IS CONTINUOUS

**THEOREM (2010 Peralta-Russo)**
NECESSARY AND SUFFICIENT CONDITIONS UNDER WHICH A DERIVATION OF A JB*-TRIPLE INTO A JORDAN TRIPLE **MODULE** IS CONTINUOUS

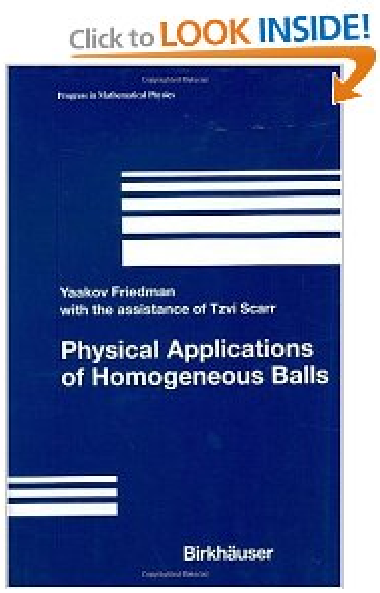
(JB*-triple and Jordan triple module are defined below)

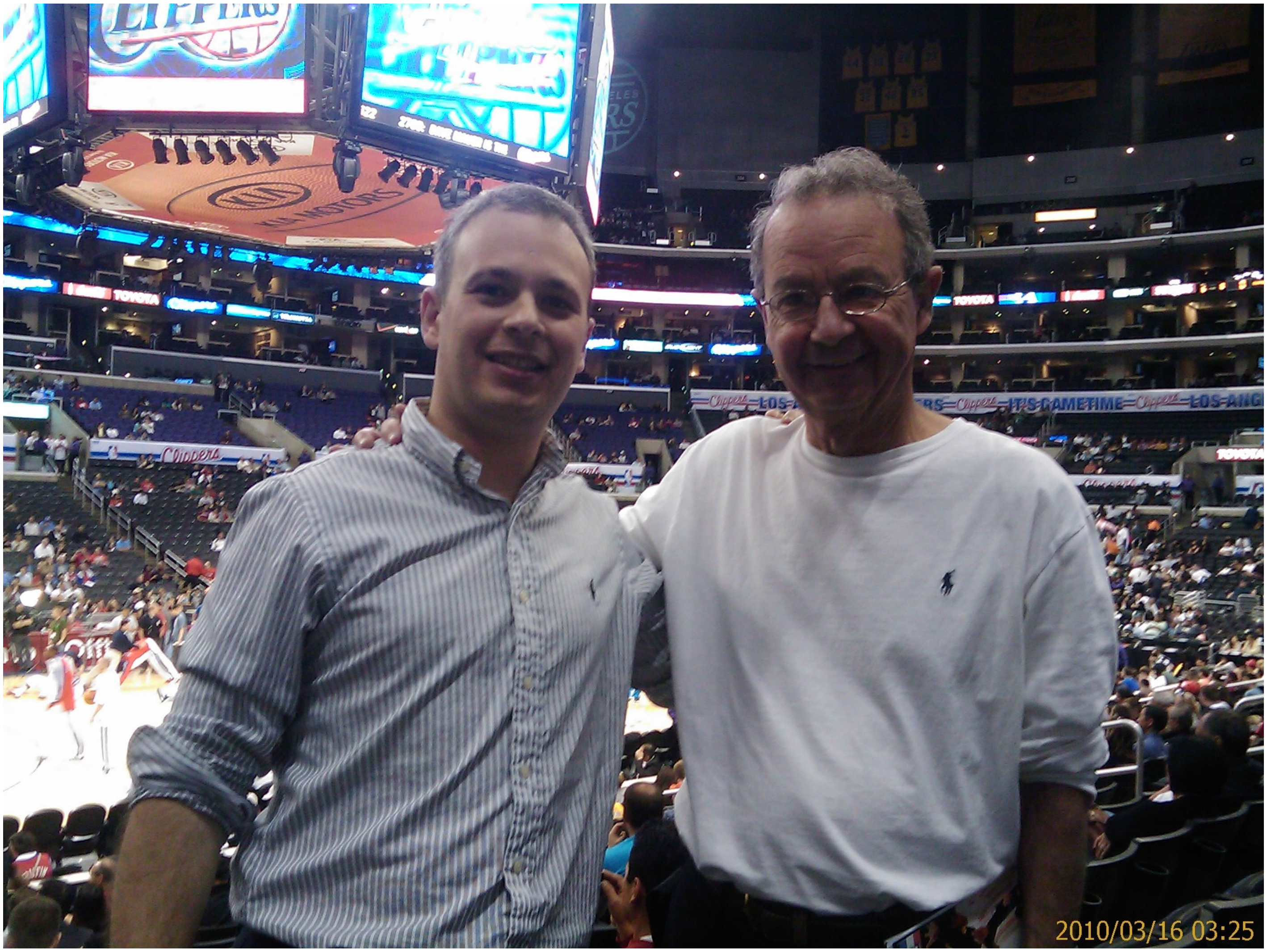
# Tom Barton (b. 1955)

Tom Barton is Senior Director for Architecture, Integration and CISO at the University of Chicago. He had similar assignments at the University of Memphis, where he was a member of the mathematics faculty before turning to administration.

**Yaakov Friedman (b. 1948)**

Yaakov Friedman is director of research at Jerusalem College of Technology.






**Antonio Peralta (b. 1974)**

**Bernard Russo (b. 1939)**

**GO LAKERS!**

**PREVIOUS INNER DERIVATION RESULTS**

(no modules here)

**THEOREM (1972 Meyberg)**

EVERY DERIVATION OF A FINITE DIMENSIONAL SEMISIMPLE JORDAN TRIPLE SYSTEM IS INNER

(Lie algebras, Lie triple systems)

**THEOREM 2002**
**(Ho-Martinez-Peralta-Russo)**

CARTAN FACTORS OF TYPE $I_{n,n}$, II (even or $\infty$), and III HAVE THE INNER DERIVATION PROPERTY

**THEOREM 2002**
**(Ho-Martinez-Peralta-Russo)**

INFINITE DIMENSIONAL CARTAN FACTORS OF TYPE $I_{m,n}, m \neq n$, and IV DO NOT HAVE THE INNER DERIVATION PROPERTY.

## SOME CONSEQUENCES FOR JB*-TRIPLES OF OUR WORK ON AUTOMATIC CONTINUITY

1. AUTOMATIC CONTINUITY OF DERIVATION ON JB*-TRIPLE

(BARTON-FRIEDMAN)

2. AUTOMATIC CONTINUITY OF DERIVATION OF JB*-TRIPLE INTO DUAL

(SUGGESTS WEAK AMENABILITY)

3. AUTOMATIC CONTINUITY OF DERIVATION OF JB*-ALGEBRA INTO A JORDAN MODULE

(HEJAZIAN-NIKNAM)

## SOME CONSEQUENCES FOR C*-ALGEBRAS OF OUR WORK ON AUTOMATIC CONTINUITY

1. AUTOMATIC CONTINUITY OF DERIVATION OF C*-ALGEBRA INTO A MODULE (RINGROSE)

2. AUTOMATIC CONTINUITY OF JORDAN DERIVATION OF C*-ALGEBRA INTO A MODULE (JOHNSON)

3. AUTOMATIC CONTINUITY OF JORDAN DERIVATION OF C*-ALGEBRA INTO A JORDAN MODULE

(HEJAZIAN-NIKNAM)

**PRELIMINARY WORK ON TERNARY WEAK AMENABILITY FOR C*-ALGEBRAS AND JB*-TRIPLES**

1. COMMUTATIVE C*-ALGEBRAS ARE TERNARY WEAKLY AMENABLE (TWA)

2. COMMUTATIVE JB*-TRIPLES ARE APPROXIMATELY WEAKLY AMENABLE

3. $B(H), K(H)$ ARE TWA IF AND ONLY IF FINITE DIMENSIONAL

4. CARTAN FACTORS $I_{n,1}$, IV ARE TWA IF AND ONLY IF FINITE DIMENSIONAL

## SAMPLE LEMMA

**The C$^*$-algebra $A = K(H)$ of all compact operators on an infinite dimensional Hilbert space $H$ is not Jordan weakly amenable.**

By the theorems of Johnson and Haagerup, we have

$$\mathcal{D}_J(A, A^*) = \mathcal{D}_b(A, A^*) = \mathcal{I}nn_b(A, A^*).$$

We shall identify $A^*$ with the trace-class operators on $H$.

Supposing that $A$ were Jordan weakly amenable, let $\psi \in A^*$ be arbitrary. Then $D_\psi$ ($= \mathrm{ad}\,\psi$) would be an inner Jordan derivation, so there would exist $\varphi_j \in A^*$ and $b_j \in A$ such that

$$D_\psi(x) = \sum_{j=1}^{n} [\varphi_j \circ (b_j \circ x) - b_j \circ (\varphi_j \circ x)]$$

for all $x \in A$.

For $x, y \in A$, a direct calculation yields

$$\psi(xy - yx) = -\frac{1}{4}\left(\sum_{j=1}^{n} b_j\varphi_j - \varphi_j b_j\right)(xy - yx).$$

It is known (Pearcy-Topping 1971) that every compact operator on a separable (which we may assume WLOG) infinite dimensional Hilbert space is a finite sum of commutators of compact operators.

By the just quoted theorem of Pearcy and Topping, every element of $K(H)$ can be written as a finite sum of commutators $[x, y] = xy - yx$ of elements $x, y$ in $K(H)$. Thus, it follows that the trace-class operator

$$\psi = -\frac{1}{4}\left(\sum_{j=1}^{n} b_j\varphi_j - \varphi_j b_j\right)$$

is a finite sum of commutators of compact and trace-class operators, and hence has trace zero. This is a contradiction, since $\psi$ was arbitrary.

## PROPOSITION

**The JB*-triple $A = M_n(C)$ is ternary weakly amenable.**

By a Proposition which is a step in the proof that commutative C*-algebras are ternary weakly amenable,

$$\mathcal{D}_t(A, A^*) = \mathcal{I}nn_b^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*),$$

so it suffices to prove that
$\mathcal{I}nn_b^*(A, A^*) \circ * \subset \mathcal{I}nn_t(A, A^*)$.

As in the proof of the Lemma, if $D \in \mathcal{I}nn_b^*(A, A^*)$ so that $Dx = \psi x - x\psi$ for some $\psi \in A^*$, then

$$\psi = [\varphi_1, b_1] - [\varphi_2, b_2] + \frac{\mathrm{Tr}\,(\psi)}{n} I,$$

where $b_1, b_2$ are self adjoint elements of $A$ and $\varphi_1$ and $\varphi_2$ are self adjoint elements of $A^*$. It is easy to see that, for each $x \in A$, we have $D(x^*) =$

$$\{\varphi_1, 2b_1, x\} - \{2b_1, \varphi_1, x\} - \{\varphi_2, 2b_2, x\} + \{2b_2, \varphi_2, x\},$$

so that $D \circ * \in \mathcal{I}nn_t(A, A^*)$.

# APPENDIX
# MAIN AUTOMATIC CONTINUITY RESULT
# (Jordan triples, Jordan triple modules, Quadratic annihilator, Separating spaces)

## Jordan triples

A complex (resp., real) *Jordan triple* is a complex (resp., real) vector space $E$ equipped with a non-trivial triple product

$$E \times E \times E \to E$$

$$(x, y, z) \mapsto \{xyz\}$$

which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one satisfying the so-called "*Jordan Identity*":

$$L(a,b)L(x,y) - L(x,y)L(a,b) =$$

$$L(L(a,b)x,y) - L(x,L(b,a)y),$$

for all $a, b, x, y$ in $E$, where $L(x,y)z := \{xyz\}$.

A JB$^*$-algebra is a complex Jordan Banach algebra $A$ equipped with an algebra involution $^*$ satisfying $\| \{a, a^*, a\} \| = \|a\|^3$, $a \in A$. (Recall that $\{a, a^*, a\} = 2(a \circ a^*) \circ a - a^2 \circ a^*)$.

A *(complex) JB$^*$-triple* is a complex Jordan Banach triple $E$ satisfying the following axioms:

(a) For each $a$ in $E$ the map $L(a, a)$ is an hermitian operator on $E$ with non negative spectrum.

(b) $\|\{a, a, a\}\| = \|a\|^3$ for all $a$ in $A$.

Every C$^*$-algebra (resp., every JB$^*$-algebra) is a JB$^*$-triple with respect to the product $\{a, b, c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ (resp., $\{a, b, c\} := (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*)$.

# Jordan triple modules

If $A$ is an associative algebra, an $A$-*bimodule* is a vector space $X$, equipped with two bilinear products $(a, x) \mapsto ax$ and $(a, x) \mapsto xa$ from $A \times X$ to $X$ satisfying the following axioms:

$$a(bx) = (ab)x, \;\; a(xb) = (ax)b, \text{ and, } (xa)b = x(ab),$$

for every $a, b \in A$ and $x \in X$.

If $J$ is a Jordan algebra, a *Jordan $J$-module* is a vector space $X$, equipped with two bilinear products $(a, x) \mapsto a \circ x$ and $(x, a) \mapsto x \circ a$ from $J \times X$ to $X$, satisfying:

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,}$$

$$2((x\circ a)\circ b)\circ a + x\circ(a^2\circ b) = 2(x\circ a)\circ(a\circ b) + (x\circ b)\circ a^2,$$

for every $a, b \in J$ and $x \in X$

If $E$ is a complex Jordan triple, a *Jordan triple $E$-module* (also called *triple $E$-module*) is a vector space $X$ equipped with three mappings

$$\begin{gathered}\{.,.,.\}_1 : X \times E \times E \to X \\ \{.,.,.\}_2 : E \times X \times E \to X \\ \{.,.,.\}_3 : E \times E \times X \to X \qquad \text{satisfying:}\end{gathered}$$

1. $\{x, a, b\}_1$ is linear in $a$ and $x$ and conjugate linear in $b$, $\{abx\}_3$ is linear in $b$ and $x$ and conjugate linear in $a$ and $\{a, x, b\}_2$ is conjugate linear in $a, b, x$

2. $\{x, b, a\}_1 = \{a, b, x\}_3$, and $\{a, x, b\}_2 = \{b, x, a\}_2$ for every $a, b \in E$ and $x \in X$.

3. Denoting by $\{.,.,.\}$ any of the products $\{.,.,.\}_1$, $\{.,.,.\}_2$ and $\{.,.,.\}_3$, the identity $\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\}$, holds whenever one of the elements $a, b, c, d, e$ is in $X$ and the rest are in $E$.

It is a little bit laborious to check that the dual space, $E^*$, of a complex (resp., real) Jordan Banach triple $E$ is a complex (resp., real) triple $E$-module with respect to the products:

$$\{a, b, \varphi\}(x) = \{\varphi, b, a\}(x) := \varphi\{b, a, x\} \qquad (1)$$

and

$$\{a, \varphi, b\}(x) := \overline{\varphi\{a, x, b\}}, \forall \varphi \in E^*, a, b, x \in E. \qquad (2)$$

For each submodule $S$ of a triple $E$-module $X$, we define its *quadratic annihilator*, $\mathrm{Ann}_E(S)$, as the set

$$\{a \in E : Q(a)(S) = \{a, S, a\} = 0\}.$$

## Separating spaces

Separating spaces have been revealed as a useful tool in results of automatic continuity.

Let $T : X \to Y$ be a linear mapping between two normed spaces. The *separating space*, $\sigma_Y(T)$, of $T$ in $Y$ is defined as the set of all $z$ in $Y$ for which there exists a sequence $(x_n) \subseteq X$ with $x_n \to 0$ and $T(x_n) \to z$.

A straightforward application of the closed graph theorem shows that a linear mapping $T$ between two Banach spaces $X$ and $Y$ is continuous if and only if $\sigma_Y(T) = \{0\}$

## Main Result

**THEOREM** Let $E$ be a complex JB*-triple, $X$ a Banach triple $E$-module, and let $\delta : E \to X$ be a triple derivation. Then $\delta$ is continuous if and only if $\mathrm{Ann}_E(\sigma_X(\delta))$ is a (norm-closed) linear subspace of $E$ and

$$\{\mathrm{Ann}_E(\sigma_X(\delta)), \mathrm{Ann}_E(\sigma_X(\delta)), \sigma_X(\delta)\} = 0.$$

**COROLLARY** Let $E$ be a real or complex JB*-triple. Then

(a) Every derivation $\delta : E \to E$ is continuous.

(b) Every derivation $\delta : E \to E^*$ is continuous.