Triple derivations on von Neumann algebras

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Outline

- Triple Modules and Derivations
- (Normal) Ternary Weak Amenability
- Sums of Commutators

Introduction

Building on earlier work of Bunce and Paschke, Haagerup showed in 1983 that every derivation of a von Neumann algebra into its predual is inner, and as a consquence that every C*-algebra is weakly amenable.

Last year, Ho, Peralta, and Russo initiated the study of ternary weak amenability in operator algebras and triples, defining triple derivations and inner triple derivations into a Jordan triple module.

Inner triple derivations on a von Neumann algebra M into its predual M_* are closely related to spans of commutators of normal functionals with elements of M. (Not really very surprising, and quite interesting)

Two consequences of that work are that **finite dimensional** von Neumann algebras and **abelian** von Neumann algebras have the property that every triple derivation into the predual is an inner triple derivation, analogous to the Haagerup result.

This rarely happens in a general von Neumann algebra, but it comes close. We prove a (triple) cohomological characterization of finite von Neumann algebras and a zero-one law for factors.

Main Result (in Words)

For any von Neumann algebra **factor**, the linear space of bounded triple derivations into the predual, modulo the norm closure of the inner triple derivations, has dimension 0 or 1. It is zero if the factor is finite (I_n, II_1), and one if the factor is infinite (I_∞, II_∞, III)

This is the beginning of a joint project with Robert Pluta. These ideas will be explored further in various contexts (C*-algebras, Jordan C*-algebras, ternary rings of operators, noncommutative L^p spaces, algebras and modules of measurable operators, . . .)

Modules

Let A be an associative algebra. Let us recall that an A-**bimodule** is a vector space X, equipped with two bilinear products $(a,x) \mapsto ax$ and $(a,x) \mapsto xa$ from $A \times X$ to X satisfying the following axioms:

$$a(bx) = (ab)x, \quad a(xb) = (ax)b, \text{ and, } (xa)b = x(ab),$$

for every $a, b \in A$ and $x \in X$.

The space $A \oplus X$ is an associative algebra with respect to the product

$$(a,x)(b,y) := (ab, ay + xb).$$

Let A be a Jordan algebra. A **Jordan** A-**module** is a vector space X, equipped with two bilinear products $(a,x) \mapsto a \circ x$ and $(x,a) \mapsto x \circ a$ from $A \times X$ to X, satisfying:

$$a \circ x = x \circ a, \quad a^2 \circ (x \circ a) = (a^2 \circ x) \circ a, \text{ and,}$$

$$2((x \circ a) \circ b) \circ a + x \circ (a^2 \circ b) = 2(x \circ a) \circ (a \circ b) + (x \circ b) \circ a^2,$$

for every $a, b \in A$ and $x \in X$.

The space $A \oplus X$ is a Jordan algebra with respect to the product

$$(a,x) \circ (b,y) := (a \circ b, a \circ y + b \circ x).$$

We must first define Jordan triple system

A complex (resp., real) **Jordan triple** is a complex (resp., real) vector space E equipped with a triple product $E \times E \times E \to E \quad , \quad (x, y, z) \mapsto \{x, y, z\}$ which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one and satisfying the so-called *"Jordan Identity"*:

$$L(a, b)L(x, y) - L(x, y)L(a, b) = L(L(a, b)x, y) - L(x, L(b, a)y),$$

for all a, b, x, y in E, where $L(x, y)z := \{x, y, z\}$.

The Jordan identity is equivalent to

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\},$$

which asserts that the map $iL(a, a)$ is a *triple derivation* (to be defined shortly).

It also shows that the span of the "multiplication" operators $L(x, y)$ is a Lie algebra. (We won't use this fact in this talk)

Jordan triple module

Let E be a complex (resp. real) Jordan triple. A **Jordan triple E-module** is a vector space X equipped with three mappings

$$\{.,.,.\}_1 : X \times E \times E \to X, \quad \{.,.,.\}_2 : E \times X \times E \to X$$

$$\text{and } \{.,.,.\}_3 : E \times E \times X \to X$$

in such a way that the space $E \oplus X$ becomes a real Jordan triple with respect to the triple product $\{a_1 + x_1, a_2 + x_2, a_3 + x_3\} =$

$$\{a_1, a_2, a_3\}_E + \{x_1, a_2, a_3\}_1 + \{a_1, x_2, a_3\}_2 + \{a_1, a_2, x_3\}_3.$$

(PS: we don't really need the subscripts on the triple products)

The Jordan identity

$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\}$, holds whenever exactly one of the elements belongs to X.

In the complex case we have the unfortunate technical requirement that $\{x, a, b\}_1$ (which $=\{b, a, x\}_3$) is linear in a and x, and **conjugate** linear in b; and $\{a, x, b\}_2$ is **conjugate** linear in a, b, x.

Every (associative) Banach A-bimodule (resp., Jordan Banach A-module) X over an associative Banach algebra A (resp., Jordan Banach algebra A) is a real Banach triple A-module (resp., A-module) with respect to the "*elementary*" product

$$\{a,b,c\} := \frac{1}{2}(abc + cba)$$

(resp., $\{a,b,c\} = (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$), where one element of a,b,c is in X and the other two are in A. In particular, this holds if $X = A$.

The dual space, E^*, of a complex (resp., real) Jordan Banach triple E is a complex (resp., real) triple E-module with respect to the products:

$$\{a,b,\varphi\}(x) = \{\varphi,b,a\}(x) := \varphi\{b,a,x\} \tag{1}$$

and

$$\{a,\varphi,b\}(x) := \overline{\varphi\{a,x,b\}}, \tag{2}$$

$\forall x \in X, a,b \in E, \varphi \in E^*$.

Derivations

Let X be a Banach A-bimodule over an (associative) Banach algebra A. A linear mapping $D : A \to X$ is said to be a **derivation** if $D(ab) = D(a)b + aD(b)$, for every a, b in A. For emphasis we call this a **binary (or associative) derivation**.

We denote the set of all continuous binary derivations from A to X by $\mathcal{D}_b(A, X)$.

When X is a Jordan Banach module over a Jordan Banach algebra A, a linear mapping $D : A \to X$ is said to be a **derivation** if $D(a \circ b) = D(a) \circ b + a \circ D(b)$, for every a, b in A. For emphasis we call this a **Jordan derivation**.

We denote the set of continuous Jordan derivations from A to X by $\mathcal{D}_J(A, X)$.

In the setting of Jordan Banach triples, a **triple** or **ternary derivation** from a (real or complex) Jordan Banach triple, E, into a Banach triple E-module, X, is a conjugate linear mapping $\delta : E \to X$ satisfying

$$\delta\{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}, \qquad (3)$$

for every a, b, c in E.

We denote the set of all continuous ternary derivations from E to X by $\mathcal{D}_t(E, X)$.

Inner derivations

Let X be a Banach A-bimodule over an associative Banach algebra A. Given x_0 in X, the mapping $D_{x_0} : A \to X$, $D_{x_0}(a) = x_0 a - a x_0$ is a bounded (associative or binary) derivation. Derivations of this form are called **inner**.

The set of all inner derivations from A to X will be denoted by $\mathcal{I}nn_b(A, X)$.

When x_0 is an element in a Jordan Banach A-module, X, over a Jordan Banach algebra, A, for each $b \in A$, the mapping $\delta_{x_0,b} : A \to X$,

$$\delta_{x_0,b}(a) := (x_0 \circ a) \circ b - (b \circ a) \circ x_0, \ (a \in A),$$

is a bounded derivation. Finite sums of derivations of this form are called **inner**.

The set of all inner Jordan derivations from A to X is denoted by $\mathcal{I}nn_J(A, X)$

Let E be a complex (resp., real) Jordan triple and let X be a triple E-module. For each $b \in E$ and each $x_0 \in X$, we conclude, via the main identity for Jordan triple modules, that the mapping $\delta = \delta(b, x_0) : E \to X$, defined by

$$\delta(a) = \delta(b, x_0)(a) := \{b, x_0, a\} - \{x_0, b, a\} \quad (a \in E), \tag{4}$$

is a ternary derivation from E into X. Finite sums of derivations of the form $\delta(b, x_0)$ are called **inner triple derivations**.

The set of all inner ternary derivations from E to X is denoted by $\mathcal{I}nn_t(E, X)$.

Ternary Weak Amenability (Ho-Peralta-R)

Proposition

Let A be a unital Banach *-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ and the Jordan product $a \circ b = (ab + ba)/2$. Then

$$\mathcal{D}_t(A, A^*) = \mathcal{D}_J^*(A, A^*) \circ * + \mathcal{I}nn_t(A, A^*).$$

Proposition

Every commutative (real or complex) C*-algebra A is **ternary weakly amenable**, that is $\mathcal{D}_t(A, A^*) = \mathcal{I}nn_t(A, A^*)$ ($\neq 0$ btw).

Proposition

The C*-algebra $A = M_n(\mathbb{C})$ is ternary weakly amenable (Hochschild 1945) and **Jordan weakly amenable** (Jacobson 1951).

Question

Is $C_0(X, M_n(\mathbb{C}))$ ternary weakly amenable?

Negative results

Proposition

The C*-algebra $A = K(H)$ of all compact operators on an infinite dimensional Hilbert space H is **not** ternary weakly amenable.

Proposition

The C*-algebra $A = B(H)$ of all bounded operators on an infinite dimensional Hilbert space H is **not** ternary weakly amenable.

Non algebra results

Theorem

Let H and K be two complex Hilbert spaces with $\dim(H) = \infty > \dim(K)$. Then the rectangular complex Cartan factor of type I, $L(H, K)$, and all its real forms are **not** ternary weakly amenable.

Theorem

Every commutative (real or complex) JB*-triple E is **approximately ternary weakly amenable**, that is, $\mathcal{I}nn_t(E, E^*)$ is a norm-dense subset of $\mathcal{D}_t(E, E^*)$.

Commutative Jordan Gelfand Theory (Kaup,Friedman-R)

Given a commutative (complex) JB*-triple E, there exists a principal $\mathbb{T}$-bundle $\Lambda = \Lambda(E)$, i.e. a locally compact Hausdorff space Λ together with a continuous mapping $\mathbb{T} \times \Lambda \to \Lambda$, $(t, \lambda) \mapsto t\lambda$ such that $s(t\lambda) = (st)\lambda$, $1\lambda = \lambda$ and $t\lambda = \lambda \Rightarrow t = 1$, satisfying that E is JB*-triple isomorphic to

$$\mathcal{C}_0^{\mathbb{T}}(\Lambda) := \{f \in \mathcal{C}_0(\Lambda) : f(t\lambda) = tf(\lambda), \forall t \in \mathbb{T}, \lambda \in \Lambda\}.$$

Normal ternary weak amenability (Pluta-R)

Corollary

Let M be a von Neumann algebra and consider the submodule $M_* \subset M^*$. Then

$$\mathcal{D}_t(M, M_*) = \mathcal{I}nn_b^*(M, M_*) \circ * + \mathcal{I}nn_t(M, M_*).$$

Note

L^∞ is ternary weakly amenable and **normally ternary weakly amenable**, that is, $\mathcal{D}_t(L^\infty, L^1) = \mathcal{I}nn_t(L^\infty, L^1)$.

Question

Is $L^\infty \otimes M_n(\mathbb{C})$ normally ternary weakly amenable?

LEMMA

If M is a von Neumann algebra, and $\psi \in M_*$ satisfies $\psi^* = -\psi$, then D_ψ is a self-adjoint mapping. Conversely, if M is properly infinite and D_ψ is self-adjoint, then $\psi^* = -\psi$.

Main Results

Theorem 1 Let M be a von Neumann algebra.

(a) If every triple derivation of M into M_* is approximated in norm by inner triple derivations, then M is finite.

(b) If M is a finite von Neumann algebra acting on a separable Hilbert space or if M is a finite factor, then every triple derivation of M into M_* is approximated in norm by inner triple derivations.

Corollary

If M acts on a separable Hilbert space, or if M is a factor, then M is finite if and only if every triple derivation of M into M_* is approximated in norm by inner triple derivations.

$$M \text{ finite } \Leftrightarrow \mathcal{D}_t(M, M_*) = \overline{\mathcal{I}nn_t(M, M_*)}$$

Problem 1

Does Theorem 1(b) hold for finite von Neumann algebras on non separable Hilbert spaces?

Theorem 2 Let M be an infinite factor

The real vector space of triple derivations of M into M_*, modulo the norm closure of the inner triple derivations, has dimension 1.

Corollary

If M is a factor, the linear space of triple derivations into the predual, modulo the norm closure of the inner triple derivations, has dimension 0 or 1: It is zero if the factor is finite; and it is 1 if the factor is infinite.

$$M \text{ infinite } \Leftrightarrow \mathcal{D}_t(M, M_*)/\overline{\mathcal{I}nn_t(M, M_*)} \sim \mathbb{R} \quad (M \text{ a factor})$$

Problem 2

Does Theorem 2 hold for all properly infinite von Neumann algebras?

Commutators in von Neumann algebras–Tools

Pearcy-Topping '69; Fack-delaHarpe '80

If M is a finite von Neumann algebra, then every element of M of central trace zero is a finite sum of commutators

Halmos '52,'54; Brown-Pearcy-Topping '68; Halpern '69

If M is properly infinite (no finite central projections), then every element of M is a finite sum of commutators

Thus for any von Neumann algebra, we have $M = Z(M) + [M, M]$, where $Z(M)$ is the center of M and $[M, M]$ is the set of finite sums of commutators in M.

Commutators in von Neumann algebras–Results

Proposition 1 Let M be a finite von Neumann algebra.

(a) If M acts on a separable Hilbert space or is a factor (hence admits a faithful normal finite trace tr), and if $\mathrm{tr}^{-1}(0) = [M_*, M]$, then M is normally ternary weakly amenable. (Extended trace)

(b) If M is a factor and M is normally ternary weakly amenable, then $\mathrm{tr}^{-1}(0) = [M_*, M]$.

Corollary

A factor of type II_1 is never normally ternary weakly amenable

Problem 3

For a factor M of type III, do we have $\{1\}_\perp = [M_*, M]$?

If M is a finite von Neumann algebra of type I_n, with $n < \infty$, we can assume

$$M = L^\infty(\Omega, \mu, M_n(\mathbb{C})) = M_n(L^\infty(\Omega, \mu)),$$

$$M_* = L^1(\Omega, \mu, M_n(\mathbb{C})_*)) = M_n(L^1(\Omega, \mu))$$

and

$$Z(M) = L^\infty(\Omega, \mu)1.$$

It is known that the center valued trace on M is given by

$$\mathrm{TR}\,(x) = \frac{1}{n}(\sum_1^n x_{ii})1 \quad , \quad \text{for } x = [x_{ij}] \in M$$

We thus define, for a finite von Neumann algebra of type I_n which has a faithful normal finite trace tr,

$$\mathrm{TR}\,(\psi) = \frac{1}{n}(\sum_1^n \psi_{ii})\mathrm{tr} \quad , \quad \text{for } \psi = [\psi_{ij}] \in M_*.$$

(a) If $\mathrm{TR}(\psi) = 0$, then ψ vanishes on the center $Z(M)$ of M.

(b) $\psi^* = -\psi$ on $Z(M)$ if and only if $\mathrm{tr}\,(\psi(\omega))$ is purely imaginary for almost every ω.

$$
\begin{aligned}
\psi(x) &= \int_\Omega \langle \psi(\omega), x(\omega)\rangle \, d\mu(\omega) = \int_\Omega \mathrm{tr}\,(\psi(\omega)x(\omega))\, d\mu(\omega) \\
&= \int_\Omega \mathrm{tr}\,([\sum_k \psi_{ik}(\omega)x_{kj}(\omega)])\, d\mu(\omega) \\
&= \int_\Omega (\sum_i \sum_k \psi_{ik}(\omega)x_{ki}(\omega))\, d\mu(\omega) \\
&= \int_\Omega (\sum_k \psi_{kk}(\omega)x_{kk}(\omega))\, d\mu(\omega) \\
&= \int_\Omega (\sum_k \psi_{kk}(\omega)) f(\omega)\, d\mu(\omega) = 0
\end{aligned}
$$

proving (a). As for (b), use $\psi(f \cdot 1) = \int_\Omega f(\omega)\mathrm{tr}\,(\psi(\omega))\, d\mu(\omega)$.

Proposition 2

Let M be a finite von Neumann algebra of type I_n with $n < \infty$, which admits a faithful normal finite trace tr (equivalently, M is countably decomposable, also called σ-finite). Then M is normally ternary weakly amenable if and only if

$$\mathrm{TR}^{-1}(0) = [M_*, M].$$

Corollary

Let M be a finite von Neumann algebra of type I_n admitting a faithful normal finite trace tr. If $\mathrm{tr}^{-1}(0) = [M_*, M]$, then M is normally ternary weakly amenable.

Problem 4

Is a finite von Neumann algebra of type I normally ternary weakly amenable? If M admits a faithful normal finite trace, is

$$\mathrm{TR}^{-1}(0) = [M_*, M]?$$

Encore

Jordan and Triple Derivations <u>in</u> von Neumann algebras

Since we now only consider derivations with the same domain and range, we contract the notation from $\mathcal{D}_b(A, X)$ to $\mathcal{D}_b(A)$, *etc.*

LEMMA

Let A be a unital Banach *-algebra equipped with the ternary product given by $\{a, b, c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ and the Jordan product $a \circ b = (ab + ba)/2$.

(a) Let D be an element in $\mathcal{I}nn_b(A)$, that is, $D = \mathrm{ad}\, a$ for some a in A. Then D is a *-derivation whenever $a^* = -a$. Conversely, if D is a *-derivation, then $a^* = -a + z$ for some z in the center of A.

(b) $\mathcal{D}_t(A) = \mathcal{D}_J^*(A) + \mathcal{I}nn_t(A)$.

PROPOSITION Let M be any von Neumann algebra.

(a) (Ho-Martinz-Peralta-Russo 2002) Every triple derivation of M is an inner triple derivation.

(b) (Upmeier 1980) Every Jordan derivation of M is an inner Jordan derivation.

PROOF

To prove (a) it suffices, by the Lemma, to show that $\mathcal{D}_J^*(M) \subset \mathcal{I}nn_t(M)$. Suppose δ is a self-adjoint Jordan derivation of M. Then δ is an associative derivation (Sinclair) and by Kadison-Sakai and the Lemma, $\delta(x) = ax - xa$ where $a^* + a = z$ is a self adjoint element of the center of M. Since $M = Z(M) + [M, M]$, where $Z(M)$ denotes the center of M, we can therefore write

$$a = z' + \sum_j [b_j + ic_j, b_j' + ic_j'],$$

where b_j, b_j', c_j, c_j' are self adjoint elements of M and $z' \in Z(M)$. It follows that

$$0 = a^* + a - z = (z')^* + z' - z + 2i \sum_j ([c_j, b_j'] + [b_j, c_j'])$$

so that $\sum_j ([c_j, b_j'] + [b_j, c_j'])$ belongs to the center of M. We now have

$$\delta = \operatorname{ad} a = \operatorname{ad} \sum_j ([b_j, b_j'] - [c_j, c_j']) \qquad (5)$$

A direct calculation shows that δ is equal to the inner triple derivation

$$\sum_j \left(L(b_j, 2b_j') - L(2b_j', b_j) - L(c_j, 2c_j') + L(2c_j', c_j) \right),$$

which proves (a).

Recall the definition

Let E be a complex (resp., real) Jordan triple and let X be a triple E-module. For each $b \in E$ and each $x_0 \in X$, we conclude, via the main identity for Jordan triple modules, that the mapping $\delta = \delta(b, x_0) : E \to X$, defined by

$$\delta(a) = \delta(b, x_0)(a) := \{b, x_0, a\} - \{x_0, b, a\} \quad (a \in E), \tag{6}$$

is a ternary derivation from E into X. Finite sums of derivations of the form $\delta(b, x_0)$ are called **inner triple derivations**.

We have just shown that a self adjoint Jordan derivation δ of M has the form (5). Then another direct calculation shows that δ is equal to the inner Jordan derivation

$$\sum_j \left(L(b_j)L(b_j') - L(b_j')L(b_j) - L(c_j)L(c_j') + L(c_j')L(c_j)\right).$$

If δ is any Jordan derivation, so are δ^* and $i\delta$, so every Jordan derivation δ is an inner Jordan derivation.

Recall the definition

When x_0 is an element in a Jordan Banach A-module, X, over a Jordan Banach algebra, A, for each $b \in A$, the mapping $\delta_{x_0,b} : A \to X$,

$$\delta_{x_0,b}(a) := (x_0 \circ a) \circ b - (b \circ a) \circ x_0, \ (a \in A),$$

is a bounded derivation. Finite sums of derivations of this form are called **inner**.

Fin